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ELEMENTS

OF

PLANE GEOMETRY

BY

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PREFACE

THE plan of this treatise on the elements of Geometry has not departed materially from that of the older texts on the subject. Instead of stating a theorem, or problem, and leading the student to discover the demonstration, or solution, by means of questions and suggestions as is done by many recent texts, each demonstration, or solution, is given in its complete, logical form and is intended as a perfect model of form and logic.

Ample opportunity for cultivating the originality and exercising the ingenuity of the student is afforded by the numerous exercises throughout the book.

Many numerical exercises testing the student's grasp of the subject, and promoting his familiarity with the facts and principles enunciated, and the careful grading of all exercises perfected by actual test in the classroom, are features of the work.

W. H. B.,
DENTON, TEXAS.
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Volume 3

PLANE GEOMETRY.

INTRODUCTION.

1. **Mathematics** is the science which investigates the properties and relations of quantity. **Geometry,** like arithmetic and algebra, is a branch of mathematics.

In **arithmetic** quantities are expressed by numbers and the various operations are performed with numbers.

In **algebra** quantities are represented by letters and the operations and relations are denoted by symbols.

Geometry treats of quantities as magnitudes having form and space relations. It is based upon a limited number of simple principles, the statement of which must be preceded by clearly enunciated definitions.

2. **Space** is assumed to extend indefinitely in all directions. Every body, whether material or immaterial, must occupy a limited portion of space.

3. A **solid,** or **volume,** is a limited portion of space with three dimensions — length, breadth, and thickness.

4. A **surface** is that which separates one volume from another. It is the common boundary of two volumes. A surface has two dimensions, length and breadth.

5. A **line** is the common boundary of two surfaces. It has one dimension, length.

6. The intersection of two lines is a **point.** A point has position, but no dimension.

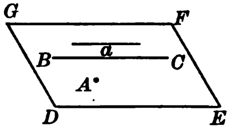
A **point** is no part of a line; a line, no part of a surface; and a surface, no part of a solid.

A moving point generates a line; a moving line generates, in general, a surface; and a moving surface generates, in general, a solid, or volume.

7. For the purpose of investigating the properties and relations of magnitudes, we make drawings showing combinations of points, lines, surfaces, and volumes. These drawings are called **geometrical figures,** but it must be understood that these are only representations of geometrical conceptions, which are, themselves, purely ideal.

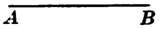
In geometry the substance composing a solid is of no importance, whatever. A block of wood or of stone is a **physical solid**; the space it occupies is a **geometric solid.** The province of geometry, therefore, is **position, form,** and **magnitude.**

8. A point is represented by a dot, and designated by a letter placed near it, as *A*. A line is represented by a mark, and designated by two letters, placed one at each end, as *BC*, or by a small letter placed near the line, as *a*. A surface is represented and designated by the lines which bound it, as *DEFG*.



LINES.

9. A **straight line** is a line which is determined by any two of its points, as *AB*.



10. From the definition of the straight line it follows:

(*a*) A straight line is the shortest path between two points.

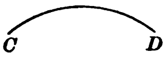
(*b*) Two straight lines can intersect in only one point, and if two straight lines have two points in common, they coincide throughout.

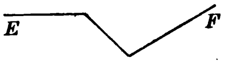
11. The word **line**, when used alone, is understood to mean a straight line of indefinite length. It is also used to denote a definite portion of such line. Where the meaning would be doubtful, a straight line of definite length is called a **line segment**, or a **sect**.

When a limited straight line, or sect, is divided into two or more parts, each part is called a **segment** of the line.

12. Two lines, or sects, are said to be **equal** when the extremities of one may be made to coincide with the extremities of the other.

13. The **mid-point** of a line is the point that divides it into two equal parts. The mid-point is said to **bisect** the line.





14. A **curved line**, or **curve**, as *CD*, is a line no part of which is straight.

15. A **broken line** is composed of a succession of straight lines. *EF* is a broken line.

SURFACES.

16. A **plane** is a surface such that the straight line, joining any two of its points, lies wholly in the surface.

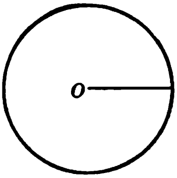
17. A **curved surface** is a surface, no portion of which, however small, is a plane.

18. A **plane figure** is a figure all points of which lie in the same plane.

19. A **rectilinear figure** is a plane figure bounded by straight lines.

20. A **curvilinear figure** is a plane figure bounded by curves.

21. A **circle** is a curve traced by a point so moving in a plane that its path is everywhere equidistant from a fixed point in the plane.



The fixed point is called the **center** of the circle.

The fixed distance from the center to the circle is called the **radius.**

The length of the circle when considered with reference to the radius was formerly called the **circumference.**

Any part of the circle is called an **arc.**

SPECIAL GEOMETRIC TERMS.

22. An **axiom** is a truth assumed to be self-evident. Axioms are common to all sciences.

23. A **geometric axiom** is an axiom peculiar to the science of geometry.

24. A **postulate** is a geometric construction so simple that its performance is readily granted.

25. A **theorem** is a statement requiring proof. The enunciation of a theorem consists of two parts: the **hypothesis,** or that which is assumed; and the **conclusion,** or that which must follow from the hypothesis.

26. A **demonstration,** or **proof,** is a train of reasoning by which the truth of a statement is established.

27. A **problem** is the statement of a proposed geometric construction.

By **construction** is meant the drawing of such straight lines, curves, or figures, as are necessary to satisfy the conditions of a problem or to aid in the demonstration of a theorem.

28. A **proposition** is either a theorem or a problem.

29. A **corollary** is a consequence, obvious, or easily deduced from a proposition or a definition.

30. GENERAL AXIOMS.

1. Things which are equal to the same thing are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. If equals are multiplied by equals, the products are equal.
5. If equals are divided by equals, the quotients are equal.
6. If equals are added to, or subtracted from, unequals, the results are unequal in the same order.
7. The whole is greater than any of its parts.
8. The whole is equal to the sum of all its parts.
9. If of three quantities, the first is greater than the second, and the second greater than the third, then the first is greater than the third.

31. GEOMETRIC AXIOMS.

1. Between two points only one straight line can be drawn.
2. Through a given point only one straight line can be drawn parallel to a given straight line. (See § 136.)
3. A geometric figure may be moved to a new position without changing its form or size.
4. Geometric figures which can be made to coincide are equal.

32. POSTULATES.

1. Between two given points a straight line can be drawn.
2. A straight line may be produced indefinitely.

3. A circle may be described with any point as center, and with any radius.

33. SYMBOLS AND ABBREVIATIONS.

ax.	. . . axiom.	⊙	. . . circle.
cor.	. . . corollary.	+	. . . plus.
def.	. . . definition.	−	. . . minus.
iden.	. . . identity.	×	. . . multiplied by.
prop.	. . . proposition.	÷, /, :	. . . divided by.
post.	. . . postulate.	=	. . . is equal to or equivalent to.
cons.	. . . construction.	∼	. . . is similar to.
hyp.	. . . hypothesis.	≅	. . . is congruent to.
rect.	. . . rectangle.	>	. . . is greater than.
rt.	. . . right.	<	. . . is less than.
st.	. . . straight.	⊥	. . . is perpendicular to, or a perpendicular.
∡	. . . angle.	//	. . . is parallel to, or a parallel.
△	. . . triangle		
▱	. . . parallelogram		

Q. E. D. (quod erat demonstrandum), which was to be proved.

Q. E. F. (quod erat faciendum), which was to be done.

Note. The foregoing are used also in the plural, as = means "are equal to," as well as "is equal to."

THE DEMONSTRATIONS OF GEOMETRY.

34. Geometry, as a science, enables us to construct figures, to discover their properties, and to determine their measurements. It is studied, not only for the acquisition of such valuable knowledge, but for training in the use of exact and concise forms of expression, and for the development of the logical powers.

35. Elementary geometry is divided into two parts: **plane geometry** and **solid geometry**.

Plane geometry treats of plane figures only. **Solid geometry**, or geometry of three dimensions, treats of figures which are not plane.

36. The demonstrations of geometry are based upon definitions, axioms, and upon theorems that have been previously proved. The usual form of proof is the **direct** demonstration, but it is often more convenient, and sometimes necessary, to use the **indirect** method, which consists in showing that an absurdity results from the assumption of the falsity of the statement, the truth of which is sought to be established.

37. The steps in the demonstration of a proposition have a logical order:

1. The **general enunciation** — with the hypothesis and the conclusion.
2. The **special enunciation** as applied to the figure used in the demonstration.
3. The **construction** or drawing of auxiliary lines, when necessary.
4. The **demonstration**, or proof.

38. When two theorems are so related that the conclusion of the first is the hypothesis of the second, and the hypothesis of the first is the conclusion of the second, each is said to be the **converse** of the other.

39. When a theorem is formed by stating the negative of both the hypothesis and the conclusion of another theorem, each is said to be the **opposite** of the other.

Example: DIRECT. If $A = B$, then $C = D$.
CONVERSE. If $C = D$, then $A = B$.
OPPOSITE. If A is not $= B$, then C is not $= D$.

40. It does not follow, as a matter of course, that either the converse of a theorem or its opposite is true.

41. If the converse and the opposite of any statement are both true, then the statement is true.

Book I.

RECTILINEAR FIGURES.

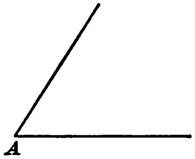
DEFINITIONS.

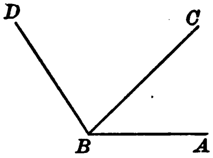
42. The divergence of two straight lines from a common point is called a **plane angle** or simply, an **angle.**

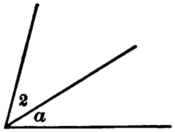
The amount of this divergence, and not the length of the lines, determines the magnitude of the angle.

43. The common point is called the **vertex,** and the lines are called the **sides** of the angle.

44. Two angles are said to be **equal,** when their sides have the same divergence, so that they may be made to coincide by applying one to the other.



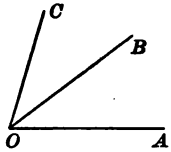




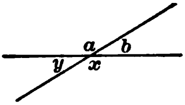
45. When an angle stands alone, it may be designated by a letter at its vertex, as the angle *A*.

When two or more angles have a common vertex, each angle is designated by three letters, the letter at the vertex being always written between the other two, as the angle *ABC* or the angle *CBD*.

It is often convenient to designate an angle by a small letter or a figure placed between its sides near the vertex, as the angle *a*, or the angle 2.

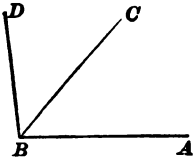


46. Two angles are called **adjacent** angles when they have the same vertex and a common side between them, as the angles *AOB* and *BOC*.

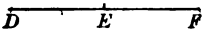


47. Two angles are called **vertical** angles when the sides of one are the prolongation of the sides of the other.

The angles *a* and *x* are vertical angles; also the angles *b* and *y*.



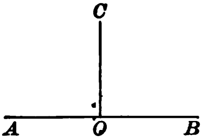
48. An angle is said to be bisected when it is divided into two equal parts. The line bisecting an angle is called its **bisector**. *BC* is the bisector of the angle *ABD*.



49. When the sides of an angle form a straight line, the angle is called a **straight** angle, as the angle *DEF*.

50. From the definition of a straight line (§ 10, *b*), it is evident that all straight angles are equal.

51. If a straight angle is bisected (§ 48), each of the two parts is called a **right** angle.



Thus, if the line *CO* meets the line *AB*, so as to divide the straight angle *BOA* into two equal angles, *BOC* and *COA*, each of these angles is a right angle.

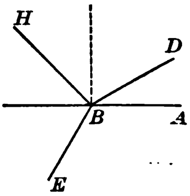
A right angle is half a straight angle, and since all straight angles are equal (§ 50), all right angles are equal.

52. When two straight lines meet so as to form right angles, each line is said to be **perpendicular** to the other. Thus, in the accompanying figure, *CO* and *AB* are perpendicular to each other.

CO is called the **perpendicular** from the point *C* to the line *AB*, and the point *O* is called the **foot** of the perpendicular *CO*.

Since all right angles are equal, it follows that only one perpendicular can be drawn to a given line at a given point in the line.

53. When the adjacent angles formed by two intersecting straight lines are unequal, each angle is called an **oblique angle,** and the lines are called **oblique lines.** Oblique angles are **acute, obtuse,** or **reflex.**

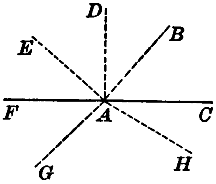


54. An **acute angle** is less than a right angle, as *ABD*.

55. An **obtuse angle** is greater than a right angle and less than a straight angle, as *ABH*.

56. A **reflex angle** is greater than a straight angle and less than two straight angles, as *ABE*.

57. Suppose the line *AB* to coincide at first with the line *AC*, and then to turn about the point *A* as a pivot, assuming in succession various positions, as *AB*, *AD*, *AE*, *AG*, etc. The moving line is said to describe the angles *CAB*, *CAD*, *CAE*, *CAF*, *CAG*, etc.



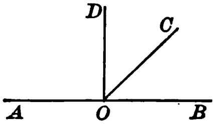
When the line *AB* has made one fourth revolution it has described a right angle; when it has made a half revolution it has described a straight angle; and when it has made a complete revolution it has described a **perigon.** A perigon is equal to the sum of two straight angles.

58. To measure an angle is to find its ratio to another angle taken as a unit of measure. The straight angle is usually taken as the unit of measure. For convenience the perigon is supposed to be divided into 360 equal parts, called **degrees**; a degree, into 60 equal parts, called **minutes**; and a minute, into 60 equal parts, called **seconds.**

A perigon contains 360 degrees; a straight angle, 180 degrees; and a right angle, 90 degrees.

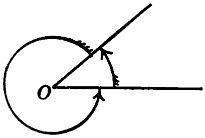
Degrees, minutes, and seconds are expressed, respectively, by the symbols, °, ′, ″.

59. Two angles whose sum is a right angle are said to be **complements** of each other, and are called **complementary** angles. Two angles whose sum is a straight angle are said to be **supplements** of each other, and are called **supplementary** angles. Two angles whose sum is a perigon are said to be **conjugates** of each other, and are called **conjugate** angles.



In the annexed figure in which *DO* is perpendicular to *AB*, the angles *BOC* and *COD* are complementary; the angles *BOC* and *COA* are supplementary.

In general, if two lines are drawn from the same point, *O*, two conjugate angles are formed. Of these the smaller is always understood by "the angle *O*" unless the contrary is stated.



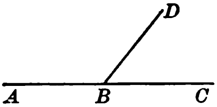
60. It follows from axiom 3 that complements of equal angles are equal; that supplements of equal angles are equal; and that conjugates of equal angles are equal.

QUESTIONS.

1. If an angle is $\frac{2}{3}$ a right angle, what is its complement?
2. If an angle is $\frac{1}{3}$ a right angle, what is its supplement?
3. If an angle is $\frac{4}{5}$ a right angle, what is its conjugate?
4. What is the complement of an angle of 30°? of 10°? of 18°?
5. If a right angle is taken from an obtuse angle, what kind of angle is left?
6. If a right angle is taken from a reflex angle, what kind of angle is left?
7. What kind of angle is the supplement of an obtuse angle?

PROPOSITION I. THEOREM.

61. *If two straight lines intersect each other, the sum of two adjacent angles thus formed is equal to a straight angle.*



Given the straight line DB intersecting the straight line AC at B

To prove $\measuredangle ABD + \measuredangle DBC =$ st. $\measuredangle$.

Proof. Because ABC is a straight line, Hyp.

$\angle ABC$ is a st. $\angle$. § 49

But $\angle ABD + \angle DBC = \angle ABC$. Ax. 8

$\therefore \angle ABD + \angle DBC = \text{st.} \angle$.

Q. E. D.

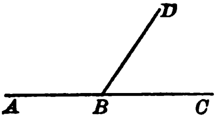
62. Cor. 1. *The sum of all the angles about a point on one side of a straight line passing through the point is equal to a straight angle.*

63. Cor. 2. *The sum of all the angles about a point is equal to two straight angles.*

Ex. 1. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.

Proposition II. Theorem.

64. *If the sum of two adjacent angles is equal to a straight angle, their exterior sides form a straight line* (converse of § 61).



Given two adjacent angles, $\angle CBD + \angle DBA = \text{st.} \angle$.

To prove *AB and BC form a straight line.*

Proof. $\angle CBD + \angle DBA = \text{st.} \angle$. Hyp.

But $\angle CBD + \angle DBA = \angle CBA$. Ax. 8

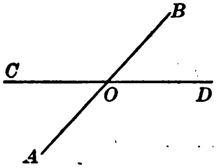
Hence $\angle CBA$ is a st. $\angle$. Ax. 1

$\therefore ABC$ is a straight line. § 49

Q. E. D.

PROPOSITION III. THEOREM.

65. *If two straight lines intersect, the vertical angles are equal.*



Given the lines AB and CD intersecting at O.

To prove $\angle AOC = \angle BOD$, *and* $\angle COB = \angle AOD$.

Proof. $\angle AOC + \angle COB =$ st. $\angle$. § 61

(If two st. lines intersect each other, the sum of two adjacent $\angle$*s thus formed = st.* $\angle$*.)*

Also $\angle BOD + \angle COB =$ st. $\angle$. § 61

$\therefore \angle AOC + \angle COB = \angle BOD + \angle COB$. Ax. 1

Subtracting the common $\angle COB$,

$\angle AOC = \angle BOD$. Ax. 3

Similarly, $\angle COB = \angle AOD$. Q. E. D.

66. COR. *If one of the four angles formed by two intersecting lines is a right angle, the other three are right angles.*

Ex. 2. How many degrees in an angle if it is double its complement ?

Ex. 3. How many degrees in an angle if its supplement is four times its complement ?

Ex. 4. How many degrees in an angle whose supplement is three times its complement ?

Ex. 5. If one of the angles made by two intersecting lines is 60°, how many degrees are there in each of the other angles?

Ex. 6. The bisectors of two vertical angles form a straight line.

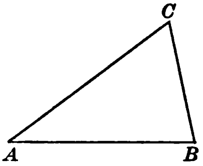
Ex. 7. The bisector of an angle bisects its vertical angle.

Ex. 8. If two lines intersect, making the sum of one pair of vertical angles equal to twice the sum of the other pair of vertical angles, what is the value in degrees of each angle?

TRIANGLES.

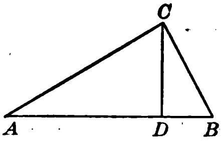
67. A **triangle** is a figure bounded by three straight lines. *ABC* is a triangle.

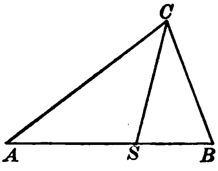
68. The **sides** of a triangle are the lines which bound it. The sum of the sides is the **perimeter.**



69. The **base** of a triangle is the side upon which it is supposed to stand. Any side may be taken as the base. The angle opposite the base is the **vertical** angle, and the angles adjacent to the base are the **base** angles.

These three angles are called the **angles** of the triangle, and the vertices of these angles are called the **vertices** of the triangle.

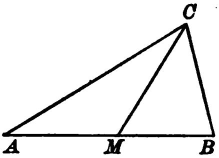




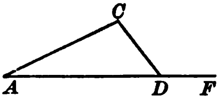
70. The **altitude** of a triangle is the perpendicular from the vertex of the vertical angle to the base or the base produced. *CD* is an altitude of the triangle *ABC.*

71. The **bisector** of a triangle is a line that bisects an angle and is produced to meet the opposite side. *CS* is a bisector.

72. A **median** of a triangle is a line from a vertex to the mid-point of the opposite side. *CM* is a median.

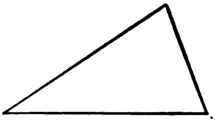


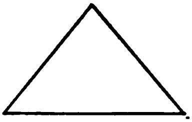
73. A triangle has three altitudes, three bisectors, and three medians.

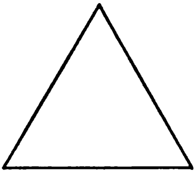


74. An **exterior angle** of a triangle is the angle formed by one side and the prolongation of another side. *CDF* is an exterior angle of the triangle *ACD*.

75. **Similar** figures are figures having the same form. **Equivalent** figures are figures having the same size. **Congruent** figures are figures having the same form and the same size.


Scalene.


Isosceles.

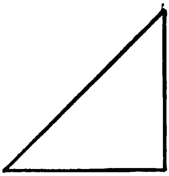

Equilateral.

76. Triangles are classified according to the relative lengths of their sides or the relative magnitudes of their angles.

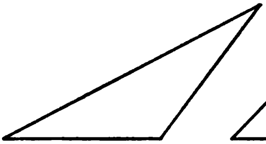
77. A **scalene** triangle has no two sides equal.

78. An **isosceles** triangle has two equal sides. The equal sides are called the **legs**, and the third side, the **base** of the triangle.

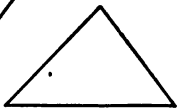
79. An **equilateral** triangle has three equal sides.



Right.



Obtuse.



Acute.

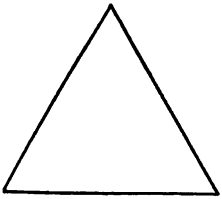
80. A **right** triangle has one right angle. The side opposite the right angle is called the **hypotenuse**, and the other two sides, the **legs**.

81. An **obtuse** triangle has one obtuse angle.

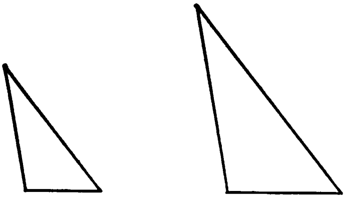
82. An **acute** triangle has three acute angles.

83. An **equiangular** triangle has its three angles equal.

84. Triangles are **mutually equilateral** if their sides are respectively equal, and **mutually equiangular** if their angles are respectively equal.



Equiangular.

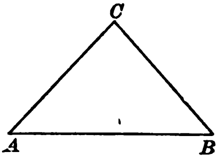


Mutually Equiangular Triangles.

85. In similar figures the corresponding vertices, sides, or angles are called **homologous**. In congruent figures the homologous parts are equal.

PROPOSITION IV. THEOREM.

86. *Any side of a triangle is less than the sum of the other two, and greater than their difference.*



Given any △ ABC.

To prove I. *any side, as* $AC < AB + BC$.

II. $AC > AB - BC$.

Proof. I. AC is a straight line. § 67

$\therefore AC < AB + BC$, § 10 (*a*)

(*a st. line is the shortest path between two points*).

II. $AB < AC + BC$,

by first part of the proof.

Subtracting BC from each side of this inequality,

$AB - BC < AC$, Ax. 6

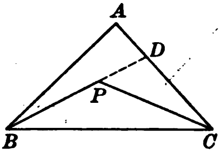
or $AC > AB - BC$. Q. E. D.

Ex. 9. The perimeter of a triangle is greater than twice any one of its sides.

QUESTION. Why is it incorrect to state as a definition that a straight line is the shortest distance between two points?

PROPOSITION V. THEOREM.

87. *The sum of any two sides of a triangle is greater than the sum of any two lines from any point within the triangle to the extremities of the third side.*



Given the lines PB and PC from any point P within the triangle ABC to the extremities of BC.

To prove $AB + AC > PB + PC.$

Proof. Produce BP to meet AC at D.

Then $AB + AD > BD.$ § 10 (*a*)

Adding DC, $AB + AD + DC > BD + DC,$ Ax. 6

or $AB + AC > BD + DC.$

But $PD + DC > PC.$ § 10 (*a*)

Adding BP, $BP + PD + DC > BP + PC,$ Ax. 6

or $BD + DC > BP + PC.$

But $AB + AC > BD + DC.$ Just proved

$\therefore AB + AC > BP + PC.$ Ax. 9

Q. E. D.

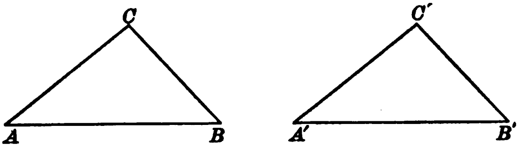
88. DEF. **Collinear points** are points that lie on the same straight line.

89. DEF. **Concurrent lines** are lines that pass through the same point.

90. DEF. The **perpendicular bisector** of a line is the line perpendicular to it at its mid-point.

PROPOSITION VI. THEOREM.

91. *Two triangles are congruent if they have two sides and the included angle of the one equal, respectively, to two sides and the included angle of the other.*



Given the $\triangle$s ABC and A'B'C', in which AB = A'B', AC = A'C', and $\measuredangle$ A = $\measuredangle$ A'.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that A shall fall on A' and AB on its equal $A'B'$.

Since $\measuredangle A = \measuredangle A'$, the side AC will fall along $A'C'$ and will coincide with $A'C'$, its equal, and C will fall on C'.

Hence BC must coincide with $B'C'$. § 10 (*b*)

$\therefore$ the $\triangle$s ABC and $A'B'C'$ coincide throughout, or are congruent. Q. E. D.

92. COR. *Two right triangles are congruent if the legs of one are equal, respectively, to the legs of the other.*

93. The Method of Superposition. To superpose one figure on another is to place one on the other so that all the parts of the one coincide with the corresponding, or homologous, parts of the other.

This method of establishing the congruence of figures is called the method of superposition. It should be employed only in fundamental propositions, and, in using it, the student should be careful to conform to the following order:

1. Beginning at a given point, place a given line upon its corresponding line.

2. Note that because of equality of angles, adjacent given lines must take the direction of corresponding lines.

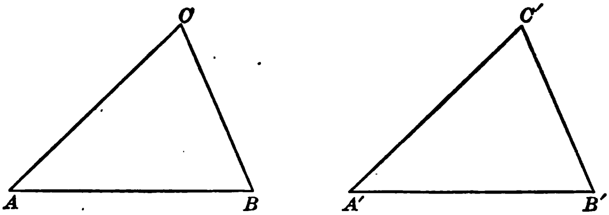
3. Because of the equality of lines, or the intersecting of coincident lines, corresponding points must coincide.

Ex. 10. How many degrees in an angle if it is one fifth its conjugate?

Ex. 11. How many degrees in the angle whose conjugate is equal to twice the sum of the supplement and the complement?

PROPOSITION VII. THEOREM.

94. *Two triangles are congruent if they have two angles and the included side of the one equal, respectively, to two angles and the included side of the other* (reciprocal of § 91).



Given △s ABC and A′B′C′, having AB = A′B′, $\measuredangle$ A = $\measuredangle$ A′, and $\measuredangle$ B = $\measuredangle$ B′.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB shall coincide with $A'B'$, the point A falling on A' and B on B'.

Now, since $\measuredangle A = \measuredangle A'$ and $\measuredangle B = \measuredangle B'$, AC will fall along $A'C'$ and BC along $B'C'$.

$\therefore$ the point C will coincide with the point C', § 10 (*b*)

(*two st. lines can intersect in only one point*).

$\therefore$ △s ABC and $A'B'C'$ coincide throughout, or are congruent. Q. E. D.

95. Cor. *Two right triangles are congruent if a leg and an adjacent acute angle of the one are equal, respectively, to a leg and an adjacent acute angle of the other.*

96. Reciprocity. Some pairs of geometric concepts have a mutual dependence upon each other, as line (determined by two points) and point (determined by two intersecting lines), etc. The names of such concepts are called **reciprocal terms**; and two theorems, as § 91 and § 94, each formed from the other by the interchange of reciprocal terms, are called **reciprocal theorems.**

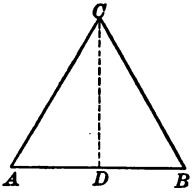
It does not follow, as a matter of course, that because a theorem is true, its reciprocal is true.

97. RECIPROCAL TERMS.

point,	line.
angle (or vertex),	side.
equiangular,	equilateral.
collinear points,	concurrent lines.

Proposition VIII. Theorem.

98. *In an isosceles triangle the angles opposite the equal sides are equal.*



Given the △ ABC, in which AC = BC.

To prove $\measuredangle A = \measuredangle B$.

Proof. Suppose CD drawn to bisect $\measuredangle BCA$.

In $\triangle$s ACD and BCD,

$CD = CD$, Iden.

$AC = BC$, Hyp.

and $\measuredangle\, ACD = \measuredangle\, BCD$. Hyp.

$\therefore \triangle\, ACD \cong \triangle\, BCD$, § 91

(having two sides and the included $\measuredangle$*s equal).*

$\therefore \measuredangle\, A = \measuredangle\, B$, § 85

(homologous $\measuredangle$*s of* $\cong \triangle$*s).* Q. E. D.

99. Cor. 1. *An equilateral triangle is equiangular.*

100. Cor. 2. *The bisector of the vertical angle of an isosceles triangle divides the triangle into two congruent triangles.*

101. Cor 3. *The bisector of the vertical angle of an isosceles triangle is the perpendicular bisector of the base.*

102. Cor. 4. *If the bisector of the vertical angle of a triangle is perpendicular to the base, the triangle is isosceles.*

Ex. 12. If the legs of an isosceles triangle are produced, the exterior angles at the base are equal.

Ex. 13. How many degrees in an angle whose conjugate is six times its complement?

Question. A moving point generates a line. Can it be said that a line is composed of points?

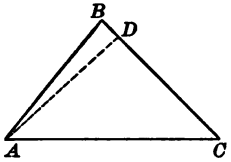
Question. What is the reciprocal of the corollary announced in § 99, an equilateral triangle is equiangular?

Question. What is the converse of the same corollary?

Question. What is the opposite of this corollary?

PROPOSITION IX. THEOREM.

103. *If two angles of a triangle are equal, the sides opposite these angles are equal and the triangle is isosceles* (reciprocal of § 98).



Given the $\triangle$ ABC in which $\measuredangle$ BAC = $\measuredangle$ C.

To prove $BA = BC$.

Proof. If BA and BC are not equal, one of them is the greater.

Suppose BC to be the greater, and from it cut off $CD = BA$, and draw DA.

Then in the $\triangle$s DAC and BAC,

$DC = BA$. Cons.

$AC = AC$. Iden.

$\measuredangle DCA = \measuredangle BAC$. Hyp.

$\therefore \triangle BAC \cong \triangle DAC$, § 91

(having two sides and the included $\measuredangle$s equal).

But $\triangle DAC$ is a part of $\triangle BAC$; that is, a part is equal to the whole, which is absurd. Ax. 7

Hence the supposition that BC is greater than BA must be abandoned.

In like manner it may be shown that BC is not less than BA.

$\therefore BA = BC$. Q. E. D.

104. COR. *An equiangular triangle is equilateral.*

LOCI.

105. If a point moves under given conditions, the line traced by the point is called a **locus**. The locus of a point in a plane is a line or a group of lines, all points of which satisfy a given condition, which is satisfied by no other points.

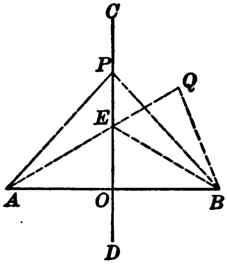
106. In proving a certain line to be the locus of a point moving under a given condition, it is necessary to prove:

1. That every point in this line satisfies the condition.
2. That no point without this line satisfies the condition, or that every point that satisfies the condition is in this line.

107. Def. The **distance** between two points is the straight line that joins them. The **distance** from a point to a given line is the perpendicular from the point to the line.

Proposition X. Theorem.

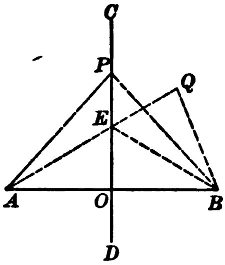
108. *The perpendicular bisector of a line is the locus of points equidistant from the extremities of the line.*



Given the line CD ⊥ AB at its mid-point O.

To prove that CD is the locus of points equidistant from A and B.

Proof. Let P be any point in CD and Q any point without CD.



Draw QA, PA, PB, and QB.

In rt. $\triangle$s POA and POB,

$PO = PO$, Iden.

$OA = OB$. Hyp.

$\therefore \triangle POA \cong \triangle POB$, § 92

(having the legs of one equal to the legs of the other).

$\therefore PA = PB$, § 85

(being homologous sides of $\cong$ $\triangle$s).

Again, since Q is not in the line CD, QA or QB must intersect CD. Let QA intersect CD at E, and draw EB.

Then, by the first part of the proof,

$$EA = EB.$$

Adding EQ to both sides,

$EA + EQ = EB + EQ$. Ax. 2

But $EB + EQ > QB$. § 10 (*a*)

$\therefore EA + EQ > QB$.

That is, $QA > QB$.

$\therefore$ CD is the locus of points equidistant from A and B. § 106

Q. E. D.

109. Cor. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.*

Ex. 14. The line joining the vertices of two isosceles triangles on the same base is the bisector of the vertical angles and the perpendicular bisector of the base.

Ex. 15. The bisectors of the base angles of an isosceles triangle are equal.

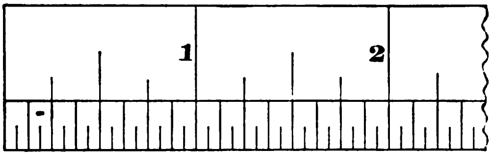
Ex. 16. The three bisectors of an equilateral triangle are equal.

CONSTRUCTIONS.

110. In elementary geometry the ruler and the compass are the only instruments allowed in the construction of figures.

The ruler is used for drawing straight lines, and the compass for laying off segments and for drawing arcs and circles.

Compass.



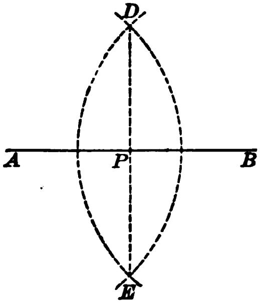
Graduated Ruler.

111. The solution of a problem consists of four parts:

1. The statement of the problem setting forth what is given and what is required.
2. The construction, or the drawing of the required parts.
3. Proof that the construction has been correctly made.
4. When necessary, the discussion of special cases or limitations.

PROPOSITION XI. PROBLEM.

112. *To bisect a given line.*



Given the line AB.

Required to bisect AB.

Construction. With *A* and *B* as centers, and with equal radii > half *AB* describe arcs intersecting at *D* and *E*. Post. 3

Draw *DE* intersecting *AB* at *P*. Post. 1

Then *P* is the required mid-point of *AB*.

Proof. *D* and *E* are by construction equidistant from *A* and *B*.

∴ *DE* is the ⊥ bisector of *AB*. § 109

Hence *AB* is bisected at *P*.

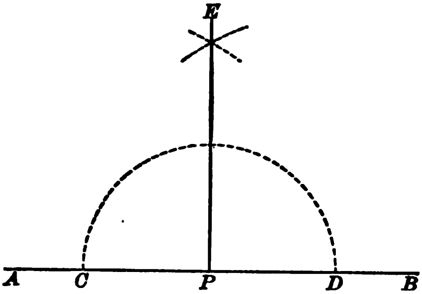
Q. E. F.

Ex. 17. To find a point in a given line equidistant from two given points.

Ex. 18. How many degrees in an angle if the sum of its conjugate, its supplement, and its complement is 570°?

PROPOSITION XII. PROBLEM.

113. *To draw a perpendicular to a given line from a given point in the line.*



Given the point P in the line AB.

Required to draw a perpendicular to AB at P.

Construction. With P as a center and any radius, PC, describe an arc cutting AB at C and D. Post. 3

With C and D as centers, with a radius greater than PC, describe arcs cutting each other at E. Post. 3

Draw PE. Post. 1

Then PE is the perpendicular required.

Proof. P and E are both equidistant from C and D. Cons.

$\therefore PE$ is the $\perp$ bisector of CD, § 109

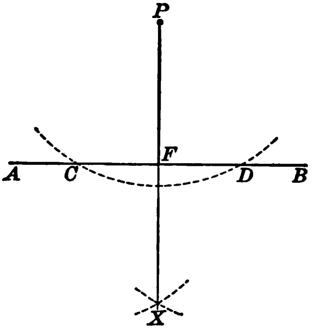
(*two points, each equidistant from the extremities of a line, determine the $\perp$ bisector of the line*).

Hence PE is perpendicular to AB. Q. E. F

NOTE. If APB is considered a straight angle, PE bisects it.

PROPOSITION XIII. PROBLEM.

114. *From a given point without a given line, to draw a perpendicular to the line.*



Given the line AB and the point P without the line.

Required to draw a ⊥ from P to AB.

Construction. With P as a center and with a radius sufficiently great, describe an arc cutting AB in C and D. Post. 3

Draw PX, the ⊥ bisector of CD, intersecting AB at F. § 112

Then PF is the ⊥ required.

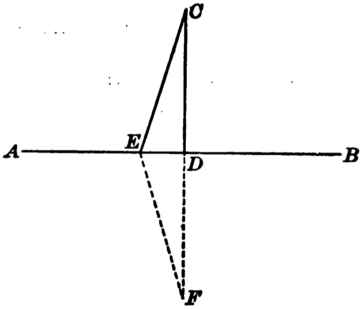
Proof. PF is ⊥ CD. Cons.

But CD and AB coincide. § 10 (*b*)

∴ PF is ⊥ AB. Q.E.F.

PROPOSITION XIV. THEOREM.

115. *Only one perpendicular can be drawn from a given external point to a given straight line.*



Given the line AB, an external point C, CD a perpendicular to AB, and CE any other line from C to AB.

To prove CE *is not* $\perp$ AB.

Proof. Produce CD to F and make $DF = CD$. Post. 2

Draw EF.

Since by construction ED is the $\perp$ bisector of CF,

$EC = EF$. § 108

$\therefore \triangle\ CEF$ is isosceles.

Hence $\measuredangle\ CED$ is half the $\measuredangle\ CEF$, § 101

(*the* $\perp$ *bisector of the base of an isosceles* $\triangle$ *bisects the vertical* $\measuredangle$).

But since CEF is not a straight line, § 10

$\measuredangle\ CEF$ is not a st. $\measuredangle$.

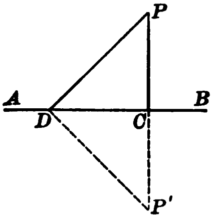
Hence $\measuredangle\ CED$ is not a right angle. § 51

$\therefore CE$ is not $\perp$ AB. § 52

Q. E. D.

PROPOSITION XV. THEOREM.

116. *The perpendicular is the shortest line that can be drawn from a given point to a given line.*



Given the straight line AB, the point P, the ⊥ PC, and any other line, as PD, from P to AB.

To prove $PC < PD$.

Proof. Produce PC to P', making $CP' = CP$, and draw DP'.

Since AB is ⊥ PP' at its mid-point C, Cons.

$$PD = P'D. \qquad \S\,108$$

$$\therefore PD + P'D = 2\,PD.$$

But $$PC + CP' = 2\,PC. \qquad \text{Cons.}$$

And $$PC + CP' < PD + DP', \qquad \S\,10$$

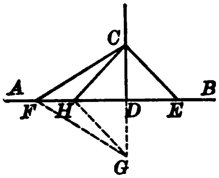
(*a straight line is the shortest path between two points*).

$$\therefore 2\,PC < 2\,PD,$$

or $$PC < PD.$$ Q. E. D.

PROPOSITION XVI. THEOREM.

117. *Two oblique lines from the same point in the perpendicular to a given line, cutting off equal segments from the foot of the perpendicular, are equal; and of two lines cutting off unequal segments from the* *foot of the perpendicular, the one cutting off the greate segment is the greater line.*



Given the line AB, the ⊥ CD, lines CE, CH, and CF from C to AB, and DE = DH and DF > DH.

To prove I. $CH = CE.$

II. $CF > CE$ or $CH.$

Proof. I. △s CDE and CDH are rt. △s by hypothesis,

and $CD = CD,$ Iden.

also $DE = DH.$ Hyp.

$\therefore \triangle CDE \cong \triangle CDH,$ § 92

(*having the legs of one = the legs of the other*).

$\therefore CH = CE,$ § 85

(*homologous sides of ≅ △s*).

II. Produce CD to G, making $DG = CD.$

Draw HG and $FG.$

Since by construction FD is the ⊥ bisector of CG,

$CH = HG,$

and $CF = FG,$ § 108

(*the ⊥ bisector of a line is the locus of points equidistant from its extremities*).

But $CF + FG > CH + HG,$ § 87

or $2\,CF > 2\,CH.$

$\therefore CF > CH$ or its equal $CE.$ Q. E. D.

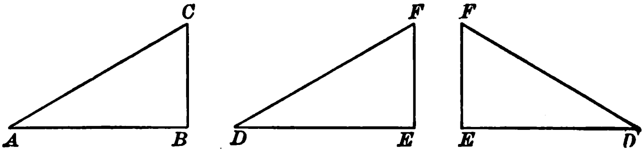
118. Cor. 1. ***Equal oblique lines from a point in a perpendicular to a given line cut off equal segments from the foot of the perpendicular.***

119. Cor. 2. ***Only two equal straight lines can be drawn from a given point to a given straight line.***

120. Cor. 3. ***Of two unequal lines from a point in a perpendicular to a given line, the greater cuts off the greater segment from the foot of the perpendicular.***

Proposition XVII. Theorem.

121. *Two right triangles are congruent if they have the hypotenuse and an acute angle of the one equal, respectively, to the hypotenuse and an acute angle of the other.*



Given the rt. $\triangle$s ABC and DEF, in which hypotenuse AC = hypotenuse DF and $\angle$ A = $\angle$ D.

To prove $\triangle ABC \cong \triangle DEF$.

Proof. Place $\triangle DEF$ on $\triangle ABC$ so that DF shall coincide with its equal AC. Then since $\angle D = \angle A$, DE will fall along AB.

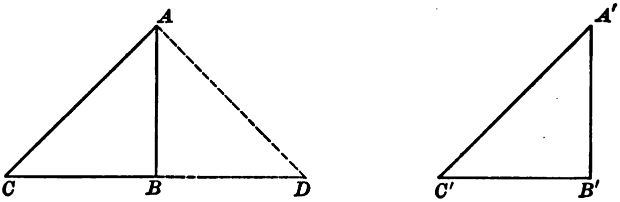
Then since $CB \perp AB$ and $FE \perp DE$, FE will coincide with CB, § 115

(only one $\perp$ can be drawn from a given point to a given straight line).

$\therefore \triangle ABC \cong \triangle DEF$. Q. E. D.

PROPOSITION XVIII. THEOREM.

122. *Two right triangles are congruent if the hypotenuse and a leg of the one are equal, respectively, to the hypotenuse and a leg of the other.*



Given the rt. △s ABC and A′B′C′, having the hypotenuse AC = the hypotenuse A′C′ and AB = A′B′.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place the $\triangle A'B'C'$ in such position that A' will fall upon A, the leg $A'B'$ upon its equal AB, and the vertex C' on the opposite side of AB from C, at D.

Since ABC and ABD are rt. △s, CBD is a straight line.

$$AC = AD. \qquad \text{Hyp.}$$

$$\therefore BC = BD, \qquad \S\ 118$$

(*equal lines from a point in a perpendicular to a given line cut off equal segments from the foot of the perpendicular*).

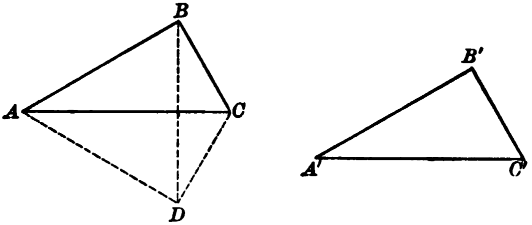
$$\therefore \triangle ABC \cong \triangle ABD, \qquad \S\ 92$$

(*having the legs of one = the legs of the other*).

$$\therefore \triangle ABC \cong \triangle A'B'C'. \qquad \text{Q. E. D.}$$

PROPOSITION XIX. THEOREM.

123. *Two triangles are congruent if they have the three sides of the one equal, respectively, to the three sides of the other.*



Given the $\triangle$s ABC and A'B'C', in which AB = A'B', AC = A'C', and BC = B'C'.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Place the $\triangle A'B'C'$ in such position that A' will fall on A, the side $A'C'$ upon its equal AC, and the vertex B' at the point D, on the opposite side of AC from B.

Draw BD.

Since $AB = AD$ and $BC = CD$, Hyp.

the $\triangle$s ABD and CBD are isosceles, and the points A and C are equidistant from the extremities of BD.

$\therefore AC$ is the $\perp$ bisector of BD. § 109

Hence $\angle BAC = \angle CAD$, § 101

(*the $\perp$ bisector of the base of an isosceles $\triangle$ bisects the vertical $\angle$*).

$\therefore \triangle ABC \cong \triangle ACD$, § 91

(*having two sides and included $\angle$ of one = two sides and included $\angle$ of the other*),

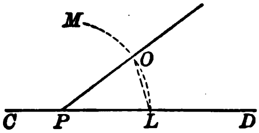
or $\triangle ABC \cong \triangle A'B'C'$. Q. E. D

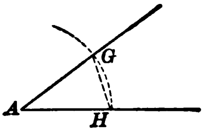
124. Cor. *If two triangles are mutually equilateral, they are mutually equiangular.*

Ex. 19. The sum of the lines from any point within a triangle to the vertices is less than the perimeter and greater than half the perimeter.

Proposition XX. Problem.

125. *At a given point in a given straight line to construct an angle equal to a given angle.*





Given the point P in the line CD, and the angle A.

Required to construct at P an angle equal to ∡ A.

Construction. With A as a center and with any radius, describe an arc cutting the sides of ∡ A at G and H. Post. 3

With P as a center and with the same radius, describe an arc LM cutting CD at L. Post. 3

Draw GH. Post. 1

With L as a center and with a radius $= GH$, describe an arc cutting the arc LM at O.

Draw PO. Post. 1

Then ∡ OPL is the angle required.

Proof. $\triangle$s POL and AGH are mutually equilateral. Cons.

$\therefore \triangle OPL \cong \triangle AGH$, § 123

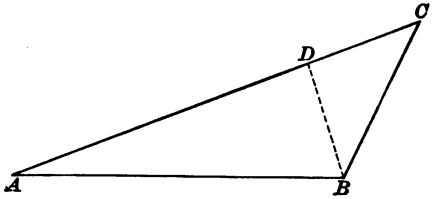
(*two* $\triangle$*s are* $\cong$ *if they have the three sides of the one equal, respectively, to the three sides of the other*).

$\therefore$ ∡ $OPL =$ ∡ A, § 85

(*homologous* ∡*s of* $\cong$ $\triangle$*s*). Q. E. F.

PROPOSITION XXI. THEOREM.

126. *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*



Given the $\triangle$ ABC, in which $\measuredangle$ CBA $>$ $\measuredangle$ C.

To prove $AC > AB$.

Proof. Draw BD meeting AC at D and making $\measuredangle CBD = \measuredangle C$. § 125

Then $\triangle BDC$ is isosceles, § 103

and $DC = DB$.

Adding AD to each side of this equation,

$$AD + DC = AD + DB,$$

or $AC = AD + DB$.

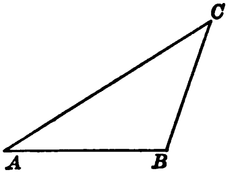
But $AD + DB > AB$. § 10

$\therefore AC > AB$. Q. E. D.

QUESTION. Geometry is often spoken of as an ideal science. Explain what is meant by this statement.

PROPOSITION XXII. THEOREM.

127. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side* (reciprocal of § 126).



Given $\triangle$ ABC, in which AC > BC.

To prove $\measuredangle B > \measuredangle A$.

Proof. If $\measuredangle B$ is not $> \measuredangle A$, it is $= \measuredangle A$ or $< \measuredangle A$.

But $\measuredangle B$ is not $= \measuredangle A$; for if $\measuredangle B = \measuredangle A$, $BC = AC$. § 103

This is contrary to the hypothesis.

And $\measuredangle B$ is not $< \measuredangle A$; for if $\measuredangle B < \measuredangle A$, $AC < BC$, § 126
which is also contrary to the hypothesis.

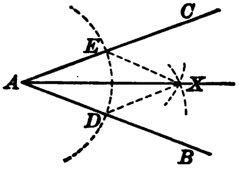
$$\therefore \measuredangle B > \measuredangle A.$$ Q. E. D.

128. NOTE. The method of demonstration used in § 127 is called **Proof by Exhaustion.** Where there are several suppositions in which one, and only one, can be true, this method consists in considering these suppositions in turn, showing the falsity of each until only one remains, which must be the true one, all others having been exhausted in the process of demonstration.

QUESTION. Is a chalk or pencil mark a geometric line?

PROPOSITION XXIII. PROBLEM.

129. *To bisect a given angle.*



Given ∡ BAC.

Required to bisect $\measuredangle BAC$.

Construction. With A as a center and with any radius AD, describe an arc cutting the sides of the given angle at D and E.

With D and E as centers and with equal radii, greater than $\frac{1}{2}DE$, describe arcs cutting each other at X.

Draw AX.

AX is the bisector required.

Proof. $\triangle$s ADX and AEX are by construction mutually equilateral and are, therefore, congruent. § 123

$$\therefore \measuredangle BAX = \measuredangle CAX, \qquad \text{§ 85}$$

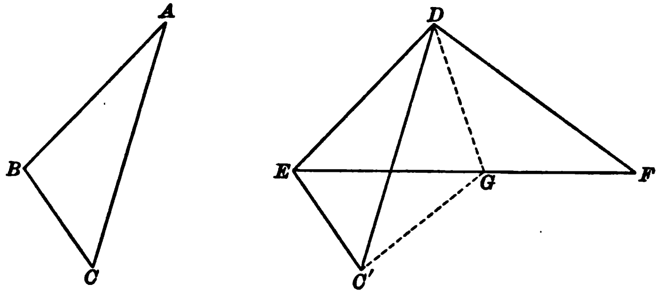
(being homologous $\measuredangle$***s of*** $\cong$ $\triangle$***s).***

Hence $\measuredangle BAC$ is bisected by the line AX. Q. E. F.

130. **NOTE. By repeating the process used in § 129, a given angle may be further divided into 4, 8, 16, 32, ⋯** 2^n **equal parts.**

PROPOSITION XXIV. THEOREM.

131. *If two triangles have two sides of the one equal, respectively, to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given $\triangle$s ABC and DEF, in which $AB = DE$, $AC = DF$, and $\angle EDF > \angle A$.

To prove $EF > BC$.

Proof. Apply $\triangle ABC$ to $\triangle DEF$, so that A falls on D, AB coincides with its equal DE, and the vertex C falls at C'.

Draw DG bisecting $\angle C'DF$, meeting EF at G. § 129

Draw GC'.

Then in $\triangle$s DGC' and DGF, $DC' = DF$, Hyp.

$DG = DG$, Iden.

and $\angle C'DG = \angle GDF$. Cons.

$\therefore \triangle DGC' \cong \triangle DGF$. § 91

$GC' = GF$, § 85

(*homologous sides of* $\cong$ $\triangle$*s*).

Adding EG to both sides of this equation,

$EG + GC' = EG + GF$. Ax. 2

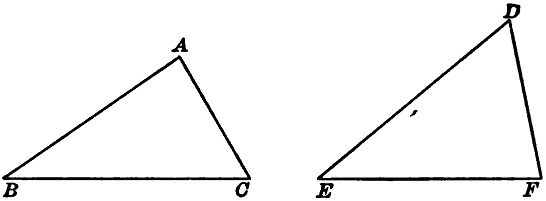
But $EG + GC' > EC'$. § 10 (*a*)

Hence $EG + GF$, or $EF > EC'$.

$\therefore EF > BC$. Q. E. D.

PROPOSITION XXV. THEOREM.

132. *If two triangles have two sides of one equal, respectively, to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given △s ABC and DEF, in which AB = DE, AC = DF, but BC > EF.

To prove $\angle A > \angle D$.

Proof. If $\angle A$ is not greater than $\angle D$, it is equal to $\angle D$, or less than $\angle D$.

If $\angle A = \angle D$, $\triangle ABC \cong \triangle DEF$, § 91

(having two sides and included $\angle$s equal),

and $BC = EF$, § 85

(homologous sides of $\cong$ $\triangle$ s).

If $\angle A < \angle D$, $BC < EF$. § 131

But both these consequences contradict the hypothesis.

Hence $\angle A$ is neither $= \angle D$ nor $< \angle D$.

$\therefore \angle A > \angle D$. Q. E. D.

133. The foregoing propositions concerning the congruence of triangles may be summarized as follows:

Two triangles are congruent if the following parts are respectively equal:

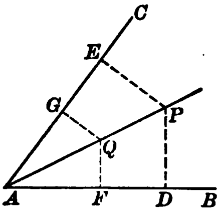
1. Two sides and the included angle.
2. Two angles and the included side.
3. Three sides.

Two right triangles are congruent if the following parts are respectively equal:

1. Two legs.
2. Hypotenuse and leg.
3. Leg and adjacent acute angle.
4. Hypotenuse and acute angle.

PROPOSITION XXVI. THEOREM.

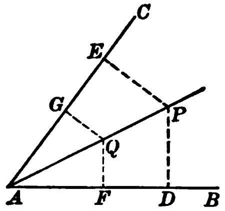
134. *The bisector of an angle is the locus of points equidistant from the sides of the angle.*



Given ∡ BAC and the point P equidistant from AB and AC.

To prove I. *AP is the bisector of ∡ BAC.*

II. *Any point Q in AP is equidistant from AB and AC.*



Proof. I. Draw $PD \perp AB$ and $PE \perp AC$. § 114

Then in rt. $\triangle$s PAD and PAE,

$PA = PA$. Iden.

$PD = PE$. Hyp.

$\therefore \triangle PAD \cong \triangle PAE$, § 122

(having hypotenuse and leg equal).

$\therefore \measuredangle PAD = \measuredangle PAE$, § 85

(homologous $\measuredangle$s of $\cong$ $\triangle$s).

Hence AP is the bisector of $\measuredangle BAC$.

II. Draw $QF \perp AB$ and $QG \perp AC$. § 114

Then in rt. $\triangle$s QAF and QAG,

$AQ = AQ$. Iden.

$\measuredangle QAF = \measuredangle QAG$. Just proved

$\therefore \triangle QAF \cong \triangle QAG$, § 121

(having hypotenuse and acute $\measuredangle$ equal).

$\therefore QF = QG$, § 85

(homologous sides of $\cong$ $\triangle$s).

Hence AP is the locus of points equidistant from AB and AC. § 106

Q. E. D.

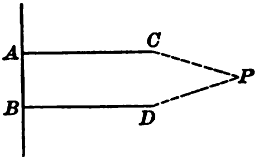
135. Cor. *Two points each equidistant from the sides of an angle determine the bisector of the angle.*

PARALLEL LINES.

136. Def. Parallel lines are straight lines, lying in the same plane, that nowhere meet, however far produced.

Proposition XXVII. Theorem.

137. *Two straight lines in the same plane perpendicular to the same straight line are parallel.*



Given the lines AC and BD, each ⊥ AB.

To prove $AC \parallel BD$.

Proof. If AC and BD are not parallel, they will meet if sufficiently produced. Suppose them to meet at some point P.

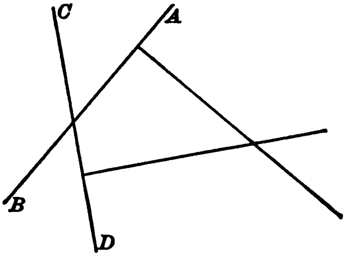
We then have two perpendiculars from the same point to the same straight line.

But this is impossible, § 115

(*only one ⊥ can be drawn from a given point to a given straight line*).

$\therefore AC$ and BD cannot meet, however far produced.

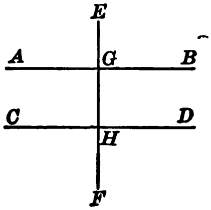
$\therefore AC \parallel BD$. Q. E. D.



138. Cor. *Two straight lines respectively perpendicular to two intersecting straight lines are not parallel.*

PROPOSITION XXVIII. THEOREM.

139. *If a straight line is perpendicular to one of two parallels, it is perpendicular to the other.*



Given AB and CD, two ‖s, and EF ⊥ CD.

To prove $EF \perp AB$.

Proof. At G, the intersection of EF and AB, conceive a line drawn ⊥ EF.

This line must be ‖ CD, § 137

(*two st. lines ⊥ the same st. line are ‖*).

But $AB \parallel CD$. Hyp.

∴ AB must coincide with the line imagined, Geom. ax. 2

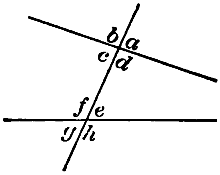
(*otherwise there would be through the point two ‖s to the same st. line*).

But the line imagined is ⊥ EF.

∴ $EF \perp AB$. Q.E.D.

140. COR. *If a straight line is parallel to one of two parallels, it is parallel to the other.*

141. DEF. A straight line cutting two or more straight lines is called a **transversal.**

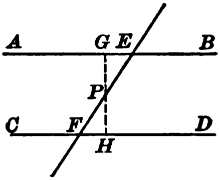


142. When a transversal cuts two lines, the eight angles thus formed are named as follows:

c, d, e, f, lying between the parallels, **interior** angles. *a, b, g, h,* lying without the parallels, **exterior** angles. *c* and *e,* or *d* and *f,* **alternate-interior** angles. *a* and *g,* or *b* and *h,* **alternate-exterior** angles. *a* and *e, b* and *f,* or *c* and *g, d* and *h,* **corresponding** angles.

PROPOSITION XXIX. THEOREM.

143. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*



Given two $\parallel$ lines, AB and CD, cut by the transversal EF.

To prove $\measuredangle AEF = \measuredangle EFD$.

Proof. Through P, the mid-point of EF, draw $PG \perp AB$ (§ 114), and produce it to meet CD at H.

Then $GH \perp CD$, § 139

(*if a st. line is $\perp$ one of two $\parallel$s, it is $\perp$ the other*).

In rt. $\triangle$s PGE and PHF,

$PE = PF$. Cons.

$\measuredangle EPG = \measuredangle FPH$, § 65

(*vertical $\measuredangle$s*).

$\therefore \triangle PGE \cong \triangle PHF$, § 121

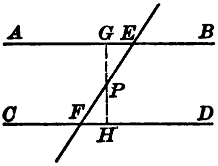
(*having hypotenuse and acute $\measuredangle$ of one = hypotenuse and acute $\measuredangle$ of the other*).

$\therefore \measuredangle AEF = \measuredangle EFD$, § 85

(*homologous $\measuredangle$s of $\cong$ $\triangle$s*). Q. E. D.

PROPOSITION XXX. THEOREM.

144. *If two straight lines are cut by a transversal, making the alternate-interior angles equal, the two straight lines are parallel* (converse of § 143).



Given the lines AB and CD cut by the transversal EF, making $\measuredangle AEF = \measuredangle DFE$.

To prove $AB \parallel CD$.

Proof. Through P, the mid-point of EF, draw $PG \perp AB$ (§ 114) and produce it to meet CD at H.

In $\triangle$s PGE and PHF,

$\measuredangle GEP = \measuredangle HFP$, Hyp.

and $\measuredangle EPG = \measuredangle FPH$, § 65

(*vertical* $\measuredangle$*s*).

Also $PE = PF$. Cons.

$\therefore \triangle PGE \cong \triangle PHF$, § 94

(*having two* $\measuredangle$*s and included side of one = two* $\measuredangle$*s and included side of the other*).

$\therefore \measuredangle PGE = \measuredangle PHF$, a right $\measuredangle$, § 85

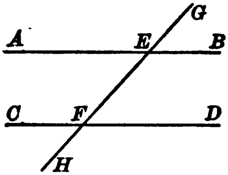
(*homologous* $\measuredangle$*s of* $\cong \triangle$*s*).

$\therefore AB \parallel CD$, § 137

(*two st. lines* $\perp$ *to the same st. line are* $\parallel$). Q. E. D

PROPOSITION XXXI. THEOREM.

145. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*



Given the ∥ lines AB and CD cut by GH at E and F, respectively.

To prove $\measuredangle DFE + \measuredangle FEB = \text{st.} \measuredangle.$

Proof. $\measuredangle AEF + \measuredangle FEB = \text{st.} \measuredangle.$ § 61

But $\measuredangle AEF = \measuredangle DFE,$ § 143

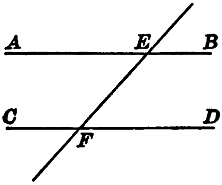
(*alt. int.* $\measuredangle$*s of* ∥ *lines*).

Substituting $\measuredangle DFE$ for its equal $\measuredangle AEF,$

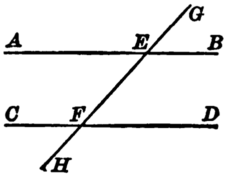
$\measuredangle DFE + \measuredangle FEB = \text{st.} \measuredangle.$ Q. E. D.

PROPOSITION XXXII. THEOREM.

146. *If two straight lines are cut by a transversal, making the interior angles on the same side of the transversal supplementary, the two lines are parallel* (converse of § 145).



Given AB and CD cut by EF, making $\measuredangle$ DFE + $\measuredangle$ FEB = st. $\measuredangle$.



To prove $AB \parallel CD$.

Proof. $\angle DFE + \angle FEB = \text{st.} \angle$. Hyp.

And $\angle AEF + \angle FEB = \text{st.} \angle$. § 61

$\therefore \angle DFE + \angle FEB = \angle AEF + \angle FEB$. Ax. 1

Taking away the common $\angle FEB$,

$\angle DFE = \angle AEF$. Ax. 3

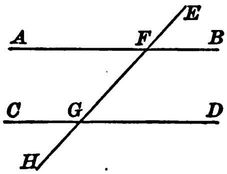
$\therefore AB \parallel CD$. § 144

Q. E. D.

147. Cor. *If two straight lines are cut by a transversal, so as to make the interior angles on the same side of the transversal together less than a straight angle, these lines will meet if sufficiently produced on that side of the transversal on which the sum of the angles is less than a straight angle.*

Proposition XXXIII. Theorem.

148. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



Given the ∥ lines AB and CD, cut by EH in F and G, respectively.

To prove $\measuredangle BFE = \measuredangle DGF$.

Proof. $\measuredangle BFE = \measuredangle AFG$, § 65

(*vertical* $\measuredangle$*s*),

and $\measuredangle DGF = \measuredangle AFG$, § 143

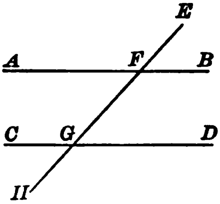
(*alt. int.* $\measuredangle$*s of* ∥ *lines*).

$\therefore \measuredangle BFE = \measuredangle DGF$. Ax. 1

Q. E. D.

PROPOSITION XXXIV. THEOREM.

149. *If two straight lines are cut by a transversal, making the corresponding angles equal, the lines are parallel* (converse of § 148).



Given AB and CD cut by EH at F and G, respectively, making $\measuredangle DGF = \measuredangle BFE$.

To prove $AB \parallel CD$.

Proof. $\measuredangle AFG = \measuredangle BFE$, § 65

(*vertical* $\measuredangle$*s*),

and $\measuredangle DGF = \measuredangle BFE$. Hyp.

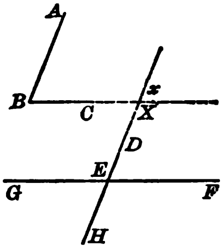
$\therefore \measuredangle AFG = \measuredangle DGF$. Ax. 1

$\therefore AB \parallel CD$. § 144

Q. E. D.

PROPOSITION XXXV. THEOREM.

150. *Two angles whose sides are, respectively, parallel are either equal or supplementary.*



Given the $\measuredangle$s at B and E, with sides BC ∥ EF and AB ∥ DH.

To prove $\measuredangle ABC = \measuredangle DEF = \measuredangle GEH$, *and* $\measuredangle$*s* ABC *and* DEG *supplementary.*

Proof. Produce, if necessary, BC and ED until they intersect at X.

Then $\measuredangle ABC = \measuredangle x$ (corresponding $\measuredangle$s),

and $\measuredangle DEF = \measuredangle x$ (corresponding $\measuredangle$s). § 148

$\therefore \measuredangle ABC = \measuredangle DEF$. Ax. 1

But $\measuredangle GEH = \measuredangle DEF$ (vertical $\measuredangle$s). § 65

$\therefore \measuredangle ABC = \measuredangle GEH$. Ax. 1

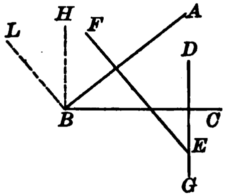
Since $\measuredangle DEF$ is supplementary to $\measuredangle DEG$ and to $\measuredangle HEF$, § 61, $\measuredangle ABC$, the equal of $\measuredangle DEF$, is supplementary to $\measuredangle DEG$ and to $\measuredangle HEF$.

Q. E. D.

151. NOTE. An angle viewed from its vertex has a right and a left side. Then, two angles having their right sides respectively parallel and also their left sides parallel, as $\measuredangle$s ABC and DEF, or $\measuredangle$s ABC and GEH, are equal, whereas if the right side of each is parallel to the left side of the other, as $\measuredangle$s ABC and FEH, they are supplementary.

PROPOSITION XXXVI. THEOREM.

152. *Two angles whose sides are, respectively, perpendicular are either equal or supplementary.*



Given the $\angle$s at B and E, with sides BA $\perp$ EF and BC $\perp$ ED.

To prove $\angle ABC = \angle DEF$, *and* $\angle ABC$ *supplementary to* $\angle FEG$.

Proof. Draw $BL \perp BA$, and $BH \perp BC$. § 113

Then $BL \parallel EF$ and $BH \parallel ED$. § 137

Since $\angle$s LBA and HBC are rt. $\angle$s, Cons.

$\angle LBH$ is the complement of $\angle HBA$,

and $\angle ABC$ is the complement of $\angle HBA$. § 59

$\therefore \angle LBH = \angle ABC$. § 60

But $\angle LBH = \angle FED$. § 151

$\therefore \angle ABC = \angle FED$. Ax. 1

Also $\angle FED$ is supplementary to $\angle FEG$. § 61

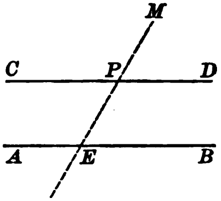
$\therefore \angle ABC$, the equal of $\angle FED$, is supplementary to $\angle FEG$. § 60

Q. E. D.

153. NOTE. Two angles whose sides are, respectively, perpendicular are equal if both angles are acute or both obtuse; they are supplementary if one is acute and the other obtuse.

PROPOSITION XXXVII. PROBLEM.

154. *Through a given point to draw a line parallel to a given line.*



Given the point P and the line AB.

Required to draw through P a line $\parallel AB$.

Construction. Through P draw a line EM intersecting AB at some point E.

At P construct the $\measuredangle MPD = \measuredangle MEB$. § 125

Then the line CD is $\parallel AB$.

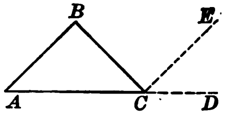
Proof. Since $\measuredangle MPD = \measuredangle MEB$, Cons.

$CD \parallel AB$, § 149

(if two st. lines are cut by a transversal, making the corresponding ∡*s equal, the lines are* ∥*).* Q. E. F.

PROPOSITION XXXVIII. THEOREM.

155. *The sum of the three angles of a triangle is equal to a straight angle.*



Given the △ ABC.

To prove $\angle A + \angle B + \angle BCA = \text{st. } \angle$.

Proof. Produce AC to D.

Draw $CE \parallel AB$. § 154

Then $\angle DCE + \angle ECB + \angle BCA = \text{st. } \angle$. § 62

But $\angle DCE = \angle A$, § 148

(*corresponding* $\angle$*s of* $\parallel$ *lines*).

And $\angle ECB = \angle B$, § 143

(*alt. int.* $\angle$*s of* $\parallel$ *lines*).

Substituting for $\angle$s DCE and ECB their equals, $\angle$s A and B, $\angle A + \angle B + \angle BCA = \text{st. } \angle$. Q. E. D.

156. Cor. 1. *The sum of any two angles of a triangle is less than a straight angle.*

157. Cor. 2. *An exterior angle of a triangle is greater than either of the non-adjacent interior angles.*

158. Cor. 3. *An exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.*

159. Cor. 4. *If two triangles have two angles of the one equal to two angles of the other, the third angle of the one is equal to the third angle of the other.*

160. Cor. 5. *Each acute angle of a right triangle is the complement of the other.*

161. Cor. 6. *There can be but one right angle or but one obtuse angle in a triangle.*

162. Cor. 7. *Each angle of an equilateral triangle is two thirds of a right angle.*

Ex. 20. If any angle of an isosceles triangle is 60°, the triangle is equilateral.

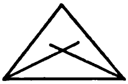
Ex. 21. The medians drawn to the legs of an isosceles triangle are equal.

Ex. 22. The three medians of an equilateral triangle are equal.

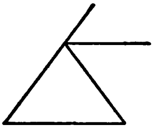
Ex. 23. In a given triangle, ABC, ∡ B is twice ∡ C, and ∡ A is three times ∡ B. How many degrees in ∡ A?

Ex. 24. How many degrees in each acute angle of an isosceles right triangle?

Ex. 25. If the vertical angle of an isosceles triangle is 40°, how many degrees in each of the base angles?



Ex. 26. If the vertical angle of an isosceles triangle is 80°, how many degrees in the angle formed by the bisectors of the base angles?



Ex. 27. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

Ex. 28. The altitudes to the legs of an isosceles triangle are equal.

Ex. 29. The three altitudes of an equilateral triangle are equal.

Ex. 30. If the altitudes to two sides of a triangle are equal, the triangle is isosceles.

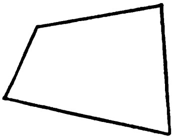
Ex. 31. If the three altitudes of a triangle are equal, the triangle is equilateral.

QUADRILATERALS.

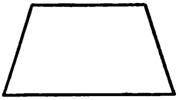
163. A **quadrilateral** is a plane figure bounded by four straight lines. The **sides** of a quadrilateral are the lines which bound it. The **perimeter** is the sum of the sides. The **angles** of a quadrilateral are those formed by the intersection of the sides, and the **vertices** are the points of their intersection. A **diagonal** is a straight line joining the vertices of any two angles that have not a common side.

164. Quadrilaterals are classified according to the number of pairs of parallel sides, and named as follows:

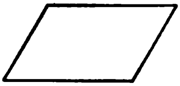
1. A **trapezium** is a quadrilateral with no pair of parallel sides.
2. A **trapezoid** is a quadrilateral which has but one pair of parallel sides. The parallel sides are called the **bases,** and the non-parallel sides are called the **legs.** The perpendicular distance between the bases is called the altitude. The **median** of a trapezoid is the straight line which joins the mid-points of the legs. An **isosceles trapezoid** has its legs equal.



Trapezium.



Trapezoid.



Parallelogram.

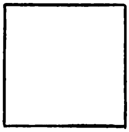
3. A **parallelogram** is a quadrilateral which has its opposite sides parallel.

165. Parallelograms are divided into species as follows:

1. A **rectangle** is a parallelogram whose angles are right angles. A **square** is an equilateral rectangle.

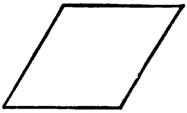
2. A **rhomboid** is an oblique angled parallelogram. A **rhombus** is an equilateral rhomboid.

Rectangle.



Square.

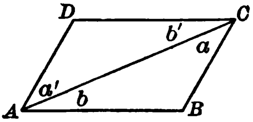
Rhomboid.



Rhombus.

Proposition XXXIX. Theorem.

166. *A diagonal divides a parallelogram into congruent triangles.*



Given the ▱ ABCD and the diagonal AC.

To prove $\triangle ABC \cong \triangle ACD$.

Proof. In $\triangle$s ABC and ACD,

$\angle a = \angle a'$, $\angle b = \angle b'$, § 143

(*alt. int.* $\angle$*s of* ∥ *lines*),

and $AC = AC$. Iden.

$\therefore \triangle ABC \cong \triangle ACD$, § 94

(*having two* $\angle$*s and included side of one equal to two* $\angle$*s and included side of the other*). Q. E. D.

167. Cor. 1. *The opposite sides and the opposite angles of a parallelogram are equal.*

For, since $\triangle$s ABC and ACD are congruent, $BC = AD$ and $AB = CD$, being homologous sides. Also $\angle B = \angle D$, homologous $\angle$s, etc.

168. COR. 2. *Two consecutive angles of a parallelogram are supplementary.*

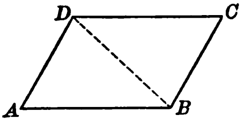
169. COR. 3. *Parallels included between parallels are equal.*

170. COR. 4. *Parallels are everywhere equidistant.*

171. COR. 5. *If one angle of a parallelogram is a right angle, the figure is a rectangle.*

PROPOSITION XL. THEOREM.

172. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral ABCD, with AB = CD and AD = BC.

To prove $ABCD$ *is a* ▱.

Proof. Draw BD.

In △s ABD and CBD,

$AD = BC,$ Hyp.

$AB = CD,$ Hyp.

and $BD = BD.$ Iden.

$\therefore \triangle ABD \cong \triangle CBD,$ § 123

(*having three sides of one = three sides of the other*).

$\therefore \measuredangle DBC = \measuredangle ADB$ and $\measuredangle DBA = \measuredangle BDC,$ § 85

(*homologous* ∡*s*).

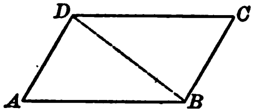
Hence $AD \parallel BC$ and $AB \parallel CD.$ § 144

$\therefore ABCD$ is a ▱. § 164 (3)

Q. E. D

PROPOSITION XLI. THEOREM.

173. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given the quadrilateral ABCD, with AB = and ∥ CD.

To prove $ABCD$ *is a* ▱.

Proof. Draw BD.

Since $AB \parallel CD$, Hyp.

$\measuredangle BDC = \measuredangle DBA$, § 143

(*alt. int.* ∡*s of* ∥ *lines*).

But $AB = CD$, Hyp.

and $BD = BD$. Iden.

$\therefore \triangle ABD \cong \triangle BCD$, § 91

(*having two sides and included* ∡*s equal*).

$\therefore \measuredangle ADB = \measuredangle DBC$, § 85

(*homologous* ∡*s*).

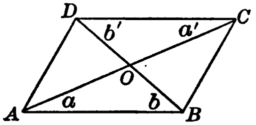
Hence $AD \parallel BC$. § 144

$\therefore ABCD$ is a ▱. § 164 (3)

Q.E.D.

PROPOSITION XLII. THEOREM.

174. *The diagonals of a parallelogram bisect each other.*



Given the ▱ ABCD and the diagonals AC and BD, intersecting at O.

To prove $AO = OC$ *and* $BO = OD$.

Proof. In $\triangle s$ AOB and DOC,

$AB = DC$, § 167

(*being opposite sides of a* ▱),

and $\measuredangle a = \measuredangle a'$; $\measuredangle b = \measuredangle b'$, § 143

(*alt. int.* $\measuredangle s$ *of* ∥ *lines*).

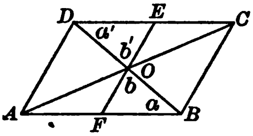
$\therefore \triangle AOB \cong \triangle DOC$, § 94

(*having two* $\measuredangle s$ *and included sides equal*).

$\therefore AO = OC$ and $BO = OD$, § 85

(*homologous sides of* $\cong \triangle s$). Q. E. D.

175. Cor. *Any line drawn through the mid-point of a diagonal and terminated in the sides of a parallelogram is bisected by the diagonal.*



Let EF be drawn through the mid-point O of the diagonal BD.

To prove $EO = OF$.

Proof. $BO = OD$. Hyp.

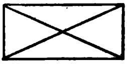
$\measuredangle a = \measuredangle a'$. § 143

$\measuredangle b = \measuredangle b'$. § 65

$\therefore \triangle OBF \cong \triangle ODE$. § 94

$\therefore EO = OF$. § 85

Ex. 32. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

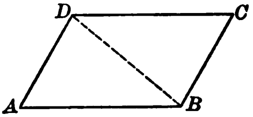


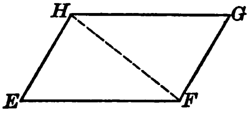
Ex. 33. The diagonals of a rectangle are equal.

Ex. 34. If the diagonals of a parallelogram are equal, the figure is a rectangle.

PROPOSITION XLIII. THEOREM.

176. *Two parallelograms are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*





Given the ▱s ABCD and EFGH, with sides AB = EF, AD = EH, and the ∡ A = ∡ E.

To prove $\square ABCD \cong \square EFGH.$

Proof. Draw BD and FH.

Since $\angle A = \angle E$, $AD = EH$ and $AB = EF$, Hyp.

$\therefore \triangle ABD \cong \triangle EFH$ (1), § 91

(having two sides and included ∡s equal).

But $\triangle ABD \cong \triangle DBC$, and $\triangle EFH \cong \triangle HFG$. § 166

$\therefore \triangle DBC \cong \triangle HFG$ (2). Ax. 1

Adding (1) and (2),

$\triangle ABD + \triangle DBC \cong \triangle EFH + \triangle HFG.$ Ax. 2

But $\triangle ABD + \triangle DBC \cong \square ABCD,$

and $\triangle EFH + \triangle HFG \cong \square EFGH.$

$\therefore \square ABCD \cong \square EFGH.$

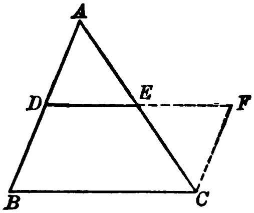
Q. E. D.

177. COR. *Two rectangles are congruent if two adjacent sides of the one are equal, respectively, to two adjacent sides of the other.*

Ex. 35. The diagonals of a rhombus are perpendicular to each other and bisect the angles of the rhombus.

PROPOSITION XLIV. THEOREM.

178. *The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the third side.*



Given the $\triangle$ ABC, and D and E the mid-points of AB and AC, respectively.

To prove $DE \parallel BC$ *and* $= \frac{1}{2} BC$.

Proof. Draw $CF \parallel BD$, meeting DE produced at F. § 154

In $\triangle$s ADE and FEC,

$AE = EC$. Hyp.

$\angle AED = \angle FEC$, § 65

$\angle ECF = \angle EAD$, § 143

$\therefore \triangle ADE \cong \triangle EFC$, § 94

$\therefore AD = CF$ and $DE = EF$. § 85

But $AD = BD$. Hyp.

$\therefore CF = BD$. Ax. 1

Since $CF \parallel BD$, Cons.

$BCFD$ is a ▱. § 173

Also, since $DF \parallel$ and $= BC$, § 164 (3) and § 167

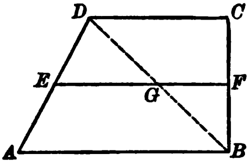
$DE \parallel BC$.

But $DE = EF = \frac{1}{2} DF$.

$\therefore DE = \frac{1}{2} BC$. Q. E. D.

179. Cor. 1. *If a line is parallel to one side of a triangle and bisects one side, it bisects the third side.*

180. Cor. 2. *The median of a trapezoid is parallel to the bases and equal to half their sum.*



Proof. Let $ABCD$ be a trapezoid and E and F the midpoints of AD and BC, respectively.

Draw BD and bisect it at G.

Draw GE and GF.

In △s ADB and DBC,

$EG \parallel$ and $= \frac{1}{2} AB$, and $GF \parallel$ and $= \frac{1}{2} CD$. § 178

But $CD \parallel AB$, § 164

hence $GF \parallel AB$. § 140

Hence EGF is a straight line. Geom. ax. 2

And $EG + GF = \frac{1}{2} AB + \frac{1}{2} CD$,

or $EF = \frac{1}{2}(AB + CD)$. Q. E. D.

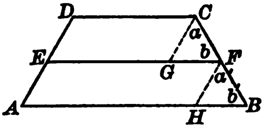
Ex. 36. The diagonals of a square are perpendicular to each other and bisect its angles.

Ex. 37. If the diagonals of a parallelogram are equal and perpendicular to each other, it is a square.

Ex. 38. Given the diagonal, to construct a square.

Ex. 39. Given the diagonals, to construct a rhombus.

181. Cor. 3. *The line drawn parallel to the bases of a trapezoid, bisecting one leg, bisects the other leg and is the median of the trapezoid.*



Given the trapezoid ABCD, F the mid-point of BC, and EF ∥ AB and CD.

Draw CG and $FH \parallel AD$.

Then $CG = DE$ and $FH = EA$. § 169

Also $\measuredangle a = \measuredangle a'$ and $\measuredangle b = \measuredangle b'$, § 148

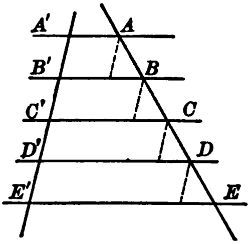
and $CF = FB$. Hyp.

$\therefore \triangle CGF \cong \triangle FHB$. § 94

Hence $CG = FH$. § 85

$\therefore DE = EA$.

182. Cor. 4. *If a series of parallels intercept equal lengths on one transversal, they intercept equal lengths on every transversal.*



Given AB = BC = CD, etc.

To prove $A'B' = B'C' = C'D'$, etc.

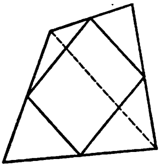
Follow the line of proof indicated in § 181.

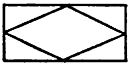
Ex. 40. The lines joining the mid-points of the sides of a triangle divide the triangle into four congruent triangles.

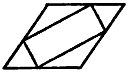
Ex. 41. Find the median of a trapezoid whose bases are 16 feet and 28 feet.

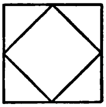
Ex. 42. Given the median of a trapezoid 12 feet and one base 20 feet, to find the other base.

Ex. 43. The lines joining the mid-points of the sides of a quadrilateral taken in order form a parallelogram (the **mid-point parallelogram** of the quadrilateral).









Ex. 44. The mid-point parallelogram of a rectangle not a square is a rhombus.

Ex. 45. The mid-point parallelogram of a rhombus is a rectangle.

Ex. 46. The mid-point parallelogram of a square is a square.

Ex. 47. The bisectors of the angles of a rhomboid form a rectangle.

Ex. 48. The bisectors of the angles of a rectangle form a square.

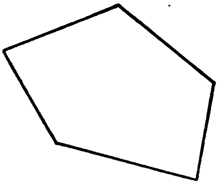
Ex. 49. The line joining the mid-points of the diagonals of a trapezoid is parallel to the bases and equal to half their difference.

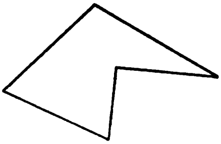
Ex. 50. The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other, and bisect the line joining the mid-points of the diagonals.

Ex. 51. The line joining the mid-points of two opposite sides of a parallelogram bisects the diagonals.

POLYGONS.

183. **Polygon** is the general name for a plane figure bounded by straight lines. The **sides** of a polygon are the lines which bound it. The **perimeter** is the sum of the sides. The **angles** of a polygon are those formed by its sides. The **vertices** are the points of intersection of the sides. A **diagonal** is a straight line joining the vertices of two angles that have not a side in common.





184. Polygons are either **convex** or **concave.** A **convex polygon** is a polygon each angle of which is less than a straight angle. A **concave polygon** is a polygon that has at least one reflex angle.

The word **polygon,** when unrestricted, means **convex polygon.**

185. An **equilateral polygon** is a polygon which has all its sides equal.

186. An **equiangular polygon** is a polygon which has all its angles equal.

187. A **regular polygon** is a polygon which is both equilateral and equiangular.

188. Two polygons are **mutually equilateral** if the sides of the one are, respectively, equal to the sides of the other when taken in the same order.

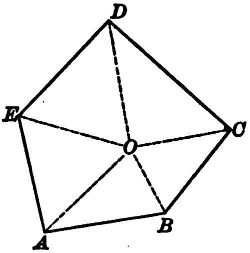
189. Two polygons are **mutually equiangular** if the angles of the one are, respectively, equal to the angles of the other when taken in the same order.

190. Two polygons may be mutually equiangular without being mutually equilateral, and, except in the case of triangles, two polygons may be mutually equilateral without being mutually equiangular.

191. A polygon of three sides is called a **triangle**; one of four sides, a **quadrilateral**; one of five sides, a **pentagon**; one of six sides, a **hexagon**; one of eight sides, an **octagon**; one of ten sides, a **decagon**; one of twelve sides a **dodecagon**; one of fifteen sides, a **pentedecagon.**

PROPOSITION XLV. THEOREM.

192. *The sum of the interior angles of a polygon is equal to as many straight angles, less two, as the polygon has sides.*



Given ABCDE ..., a polygon of n sides.

To prove $\measuredangle A + \measuredangle B + \measuredangle C + \measuredangle D$, *etc.* $= (n-2)$ *st.* ∡*s.*

Proof. From any interior point O draw lines to each vertex, thus dividing the polygon into n triangles.

The sum of the ∡s of these n △s $= n$ st. ∡s, § 155

(the sum of the ∡s of a △ = a st. ∡).

But the ∡s about the point O are not angles of the polygon.
The sum of the ∡s about $O = 2$ st. ∡s. § 63
∴ the sum of the ∡s of the polygon $= n$ st. ∡s $- 2$ st. ∡s,

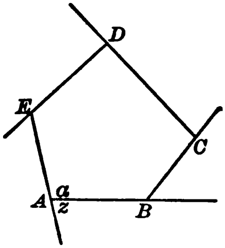
$= (n-2)$ st. ∡s. Q. E. D.

193. COR. *The sum of the angles of a quadrilateral is two straight angles, and each angle of an equiangular polygon of n sides is* $\frac{n-2}{n}$ *st.* ∡*s.*

Ex. 52. The sum of the angles of a pentagon is equal to how many right angles?

PROPOSITION XLVI. THEOREM.

194. *The sum of the exterior angles of a polygon, formed by successively producing the sides, is equal to two straight angles.*



Given ABCDE ···, a polygon of n sides, each side successively produced.

To prove the sum of the exterior ∡s = 2 st. ∡s.

Proof. Each interior ∡, as a, added to the adjacent exterior ∡, as z, = st. ∡. § 61

But the sum of the interior angles = $(n-2)$ st. ∡s. § 192

∴ the sum of the exterior ∡s = n st. ∡s − $(n-2)$ st. ∡s
= 2 st. ∡s. Q.E.D.

195. COR. *In an equiangular polygon of n sides each exterior angle is equal to* $\frac{2 \text{ st. } \measuredangle\text{s}}{n}$.

Ex. 53. How many sides has a polygon the sum of whose angles is equal to four straight angles?

Ex. 54. How many degrees in each angle of an equiangular hexagon?

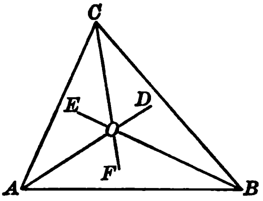
Ex. 55. How many sides has a regular polygon if each angle is four fifths of a straight angle?

Ex. 56. How many diagonals has a polygon of n sides?

Ex. 57. How many diagonals has a polygon of fourteen sides?

PROPOSITION XLVII. THEOREM.

196. *The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.*



Given AD, BE, and CF, the bisectors of △ ABC.

To prove AD, BE, and CF meet in a point equidistant from AB, BC, and AC.

Proof. Since AD is the bisector of ∡ A, it is the locus of points equidistant from AB and AC. § 134

And since CF is the bisector of ∡ C, it is the locus of points equidistant from AC and BC. § 134

∴ O, the intersection of these loci, is equidistant from AB and BC. Ax. 1

Hence O lies on BE, the bisector of ∡ B. § 134

∴ the bisectors are concurrent at O equidistant from the sides.

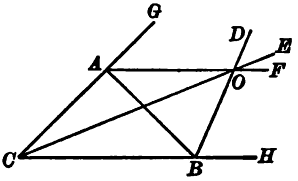
Q. E. D.

197. DEF. The point in which the bisectors of a triangle meet is called the **in-center** of the triangle.

Ex. 58. The bisector of the exterior angle at one vertex of a triangle and the bisector of the interior angle at a second vertex form an angle equal to half the interior angle at the third vertex.

PROPOSITION XLVIII. THEOREM.

198. *The bisectors of the exterior angles at two vertices of a triangle and the bisector of the interior angle at the third vertex are concurrent.*



Given △ ABC with the sides CA and CB produced to G and H, and AF, BD, and CE, the bisectors of ∡s GAB, ABH, and ACB, respectively.

To prove *AF, BD, and CE concurrent.*

Proof. *AF* is the locus of points equidistant from *GA* and *AB*. § 134

BD is the locus of points equidistant from *AB* and *BH*. § 134

Therefore, *O*, the intersection of these loci, is equidistant from *GC* and *CH*.

Hence *O* lies on *CE*. § 134

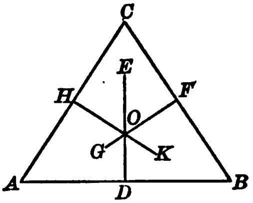
∴ *AF*, *BD*, and *CE* are concurrent at *O*. Q. E. D.

199. DEF. The three points of concurrence of the bisectors of the interior angles of a triangle and the bisectors of the non-adjacent exterior angles, respectively, are called the **excenters** of the triangle.

Ex. 59. If four rods are hinged together to form a quadrilateral, can the rods be moved so as to change the angles of the quadrilateral? How would it be with three rods hinged together to form a triangle?

PROPOSITION XLIX. THEOREM.

200. *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle.*



Given DE, FG, and HK, the ⊥ bisectors of the sides of △ ABC.

To prove DE, FG, and HK are concurrent at O, a point equidistant from A, B, and C.

Proof. FG and HK must intersect at some point, O. § 138

Since FG is the locus of points equidistant from B and C, and HK the locus of points equidistant from A and C, § 108

O, the intersection of these loci, is equidistant from A and B.

∴ O lies in the ⊥ bisector DE. § 108

∴ DE, FG, and HK are concurrent at a point equidistant from A, B, and C. Q. E. D.

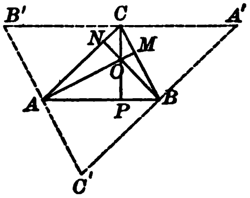
201. COR. *The point of concurrence of any two perpendicular bisectors of the three lines joining any three points is equidistant from the three points.*

202. DEF. The point of concurrence of the perpendicular bisectors of the sides of a triangle is called the **circumcenter** of the triangle.

Ex. 60. How many independent parts determine a triangle? A quadrilateral? A pentagon? A polygon of n sides?

PROPOSITION L. THEOREM.

203. *The altitudes of a triangle are concurrent.*



Given the △ ABC.

To prove the altitudes AM, BN, and CP are concurrent.

Proof. Through A draw $B'C' \parallel BC$; through B draw $A'C' \parallel AC$ and through C draw $A'B' \parallel AB$. § 154

Then $ABCB'$ and $ABA'C$ are ▱s. § 164 (3)

Hence $A'C = AB$ and $B'C = AB$. § 167

$\therefore A'C = B'C$, and C is the mid-point of $A'B'$.

Similarly, A and B are the mid-points, respectively, of $B'C'$ and $A'C'$.

But $CP \perp A'B'$, $AM \perp B'C'$, and $BN \perp A'C'$. § 139

$\therefore AM$, BN, and CP are the $\perp$ bisectors of the sides of $\triangle A'B'C'$,

$\therefore AM$, BN, and CP are concurrent. § 200

Q. E. D.

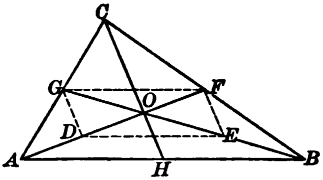
204. The point of concurrence of the altitudes of a triangle is called the **orthocenter** of the triangle.

Ex. 61. If four points are so arranged that one of them is the orthocenter of the triangle whose vertices are the other three, then each of the four points is the orthocenter of the triangle whose vertices are the other three.

Ex. 62. The vertex of the right angle is the orthocenter of a right triangle.

PROPOSITION LI. THEOREM.

205. *The medians of a triangle are concurrent in a point two thirds the distance from each vertex to the mid-point of the opposite side.*



Given the △ ABC.

To prove the medians AF, BG, and CH are concurrent in a point two thirds the distance from A to mid-point of BC, etc.

Proof. Let AF and BG intersect at O.

Bisect AO at D and BO at E. § 112

Draw DE, GF, GD, and FE.

DE and GF are each ∥ AB and $= \frac{1}{2} AB$. § 178

∴ DE and GF are = and ∥ and $DEFG$ is a ▱. § 173

∴ DF and EG bisect each other at O. § 174

Hence $BE = EO = OG$.

Or AF and BG intersect at O so that $BO = \frac{2}{3} BG$.

Similarly it is shown that CH intersects BG at a point whose distance from $B = \frac{2}{3} BG$, that is, at O.

∴ AF, BG, and CH are concurrent at O. Q. E. D.

206. The point of concurrence of the medians of a triangle is called the **centroid** of the triangle.

Ex. 63. If two medians of a triangle are equal, the sides to which they are drawn are equal and the triangle is isosceles.

Ex. 64. If the three medians of a triangle are equal, the triangle is equilateral.

SUGGESTIONS CONCERNING THE SOLUTION OF PROBLEMS.

207. No definite rule can be given for demonstrating theorems or solving problems. Were it possible to reduce the solution of all questions to fixed rules, the subject would thereby be reduced to the application of a few formulas, and thus would be destroyed the disciplinary value of the study of geometry, the object of which is not so much the discovery of facts as the acquisition of the power to discover facts.

While it is not possible to state a general method for the solution of questions, a few suggestions tending to systematize the methods of attack are not improper.

1. Figures should be drawn as accurately as possible and as general as possible, *i.e.* a figure relating to a polygon should not be a triangle; one relating to a quadrilateral should not be a square, or even a parallelogram; one relating to a triangle should not be a right or an isosceles triangle. Again, in comparison of figures, for example, two squares, their sides or other parts should be represented by lines and not by numbers, either abstract or concrete; as 5 feet and 7 feet, for the assumption that the ratio of the sides of the two squares is a rational quantity, as $\frac{5}{7}$, makes the problem or the solution a special case.

2. Be careful of the distinction between what is given and what is required with reference to the letters of the figure constructed.

3. Assume the theorem to be true, and discover what follows this assumption; then see if this conclusion can be reached without such assumption, and if so, reverse the process.

4. If the assumption of the truth of the theorem and the conclusions drawn from such assumption fails to disclose the method of procedure, assume that the statement is false and then try to show that this assumption leads to an absurdity.

Ex. 65. What is the sum of the interior angles of a polygon of 10 sides?

Ex. 66. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

Ex. 67. The incenter, the circumcenter, the orthocenter, and the centroid of an equilateral triangle are together at the same point, called the center of the equilateral triangle.

Ex. 68. The bases of an isosceles trapezoid make equal angles with the legs.

Ex. 69. If the angles at one base of a trapezoid are equal, the other angles are equal and the trapezoid is isosceles.

Ex. 70. The diagonals of an isosceles trapezoid are equal.

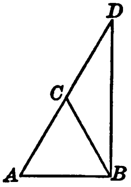
Ex. 71. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

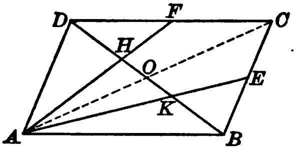
Ex. 72. The altitude of an isosceles triangle bisects the base.

Ex. 73. If the diagonals of a quadrilateral are equal, the lines joining the mid-points of its opposite sides are perpendicular to each other.

Ex. 74. The median to any side of a triangle is less than half the sum of the other two sides.

Ex. 75. The sum of the medians of a triangle is less than its perimeter.





Ex. 76. If one of the legs of an isosceles triangle is prolonged its own length, the line from the extremity of the leg produced to the nearer end of the base is perpendicular to the base.

Ex. 77. Lines from a vertex of a parallelogram to the mid-points of the non-adjacent sides trisect a diagonal.

Ex. 78. Two isosceles triangles are congruent if one side and one angle of the one are equal, respectively, to the corresponding side and angle of the other.

Ex. 79. The sum of the diagonals of a quadrilateral is less than its perimeter.

Ex. 80. If each exterior angle of a polygon is 15°, how many sides has the polygon?

Ex. 81. How many sides has a polygon if the sum of its interior angles is equal to four times the sum of its exterior angles?

Ex. 82. If two parallelograms have an angle of the one equal to an angle of the other, they are mutually equiangular.

Ex. 83. A parallelogram is divided into two congruent parts by a line drawn through the intersection of its diagonals.

Ex. 84. To construct an isosceles triangle, having given the base and the vertical angle.

Ex. 85. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

Ex. 86. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

Ex. 87. The bisectors of two exterior angles of a triangle form an angle equal to half the third exterior angle.

Ex. 88. Two triangles are congruent if two sides and the median to the third side of the one are equal, respectively, to the corresponding sides and median of the other.

Ex. 89. Two triangles are congruent if one side and the medians to the other sides in one triangle are equal, respectively, to the corresponding side and medians of the other triangle.

Ex. 90. Two triangles are congruent if the three medians of the one are equal, respectively, to the three medians of the other.

Ex. 91. Two triangles are congruent if two sides and the altitude to the third side of the one are equal, respectively, to the corresponding sides and altitude of the other.

Ex. 92. Two triangles are congruent if one side and the altitudes to the other two sides of the one are equal, respectively, to the corresponding side and altitudes of the other.

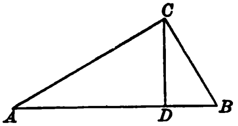
Ex. 93. Two triangles are congruent if they have a side, the median, and the altitude to that side in the one equal, respectively, to the corresponding side, median, and altitude in the other.

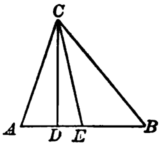
Ex. 94. Two parallelograms are congruent if two adjacent sides and a diagonal of the one are equal, respectively, to the corresponding sides and diagonal of the other.

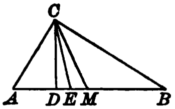
Ex. 95. Two parallelograms are congruent if the diagonals of the one are equal, respectively, to the diagonals of the other and form equal angles with each other.

Ex. 96. Two rectangles are congruent if a side and a diagonal of the one are equal, respectively, to a side and a diagonal of the other.

Ex. 97. The angle formed by the bisectors of any two consecutive angles of a quadrilateral is equal to half the sum of the other two angles.







Ex. 98. An altitude of a triangle divides an angle into two parts whose difference is equal to the difference of the other two angles of the triangle.

Ex. 99. The angle between the altitude and the bisector from the same vertex of a triangle is half the difference of the other two angles of the triangle.

Ex. 100. The bisector of the right angle of a right triangle is also the bisector of the angle formed by the altitude and the median from the vertex of the right angle.

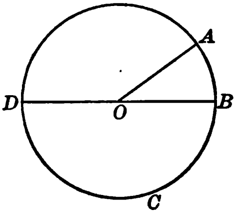
BOOK II.

THE CIRCLE.

208. A **circle** is a curve traced by a point so moving in a plane that its path is everywhere equidistant from a fixed point in the plane. The fixed point is called the **center.**

209. A **radius** is any straight line drawn from the center to the circle.

210. A **diameter** is a straight line drawn through the center and terminating in the circle.



211. The **area of a circle** is the portion of a plane bounded by the circle.

A circle is indicated by a letter at the center, as the circle O; or by two or more points on the circle, as the circle ABC.

212. Circles that are congruent are called **equal** circles.

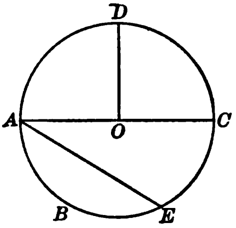
The following properties are immediately inferred as corollaries from the foregoing definitions:

213. Radii of the same circle or of equal circles are equal.

214. Diameters of the same circle or of equal circles are equal.

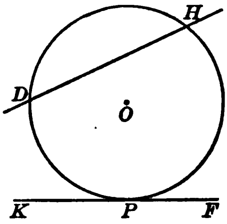
215. Circles having equal radii or equal diameters are equal

216. An **arc** is any part of the circle, as AB. If the arc is half the circle, it is called a **semicircle**, as ABC; if equal to one fourth the circle, it is called a **quadrant**, as DA.



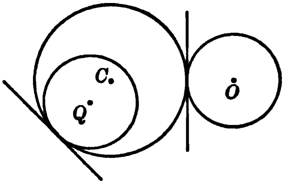
217. A **chord** is a straight line joining the extremities of an arc. AE is a chord. Every chord subtends two arcs, a **major arc** and a **minor arc**. The major arc is greater and a minor arc less than a semicircle. The sum of the two is a circle. The word *arc*, when unrestricted, is used to denote the minor arc. Two arcs whose sum is a circle are called **conjugate arcs**.

218. A **secant** is a straight line that intersects a circle in two points, as DH.



219. A **tangent** is a straight line that touches a circle at one point only; KF is a tangent.

220. The point in which a tangent touches a circle is called the **point of tangency** or **point of contact**.

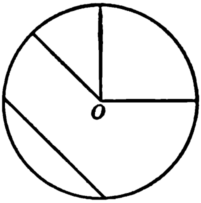


221. Two circles are tangent to each other when both are tangent to the same straight line at the same point. They are tangent internally or externally according as one circle lies within or without the other.

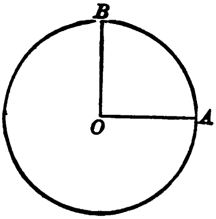
222. A **sector** is a portion of the area of a circle bounded by two radii and the intercepted arc.

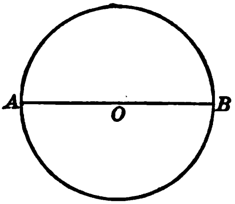
223. A **segment** is a portion of the area of a circle bounded by an arc and its subtending chord.

224. Congruent sectors and congruent segments are called **equal sectors** and **equal segments.**



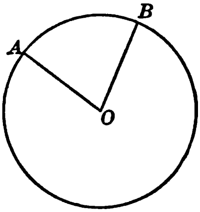
225. A **quadrant sector** is a sector that is one fourth the area of a circle.



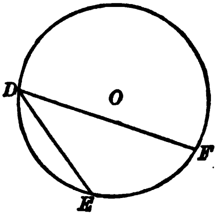


226. A **semicircle sector** is a sector that is half the area of a circle.

227. A **central angle** is an angle formed by two radii, as the angle *AOB*.



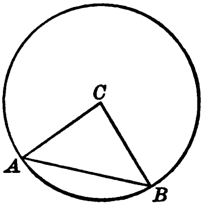
228. An **inscribed angle** is an angle whose vertex is on the circle and whose sides are chords, as angle *EDF*.

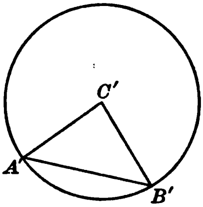


229. An angle is said to be **inscribed in a segment** if its vertex lies in the arc and the sides of the angle pass through the extremities of the arc.

PROPOSITION I. THEOREM.

230. *In the same circle, or in equal circles, equal central angles intercept equal chords and equal arcs.*





Given the equal $\odot$s C and C′, in which the central $\angle$ C = the central $\angle$ C′.

To prove

chord $AB =$ chord $A'B'$, and arc $AB =$ arc $A'B'$.

Proof. Place the $\odot$ C on the $\odot$ C' so that the center C shall fall upon the center C', and the point A on the point A'. Then since the two circles are equal, they will coincide throughout.

Because $\angle C = \angle C'$, the radius CB will fall upon $C'B'$ and the point B will fall on the point B'.

∴ chord AB will coincide with chord $A'B'$ and be equal to it, and arc AB will coincide with arc $A'B'$ and be equal to it. § 208

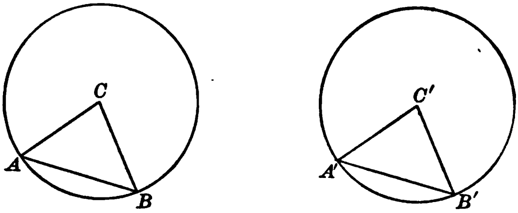
Q. E. D.

231. COR. 1. *In the same circle, or in equal circles, sectors with equal central angles are equal.*

232. COR. 2. *A diameter divides the circle and its area into two equal parts.*

PROPOSITION II. THEOREM.

233. *In the same circle, or in equal circles, equal arcs are subtended by equal chords and intercepted by equal central angles.*



Given $\odot C = \odot C'$ **and arc** $AB =$ **arc** $A'B'$.

To prove chord $AB =$ *chord* $A'B'$, *and* $\angle C = \angle C'$.

Proof. Place $\odot C$ on $\odot C'$ so that the center C shall fall on the center C', and the point A on the point A'; then since the two circles are equal, they coincide.

Then since arc $AB =$ arc $A'B'$, Hyp.

the point B will fall on the point B' and the chord AB will coincide with the chord $A'B'$.

$\therefore$ chord $AB =$ chord $A'B'$.

Again, in $\triangle$s ABC and $A'B'C'$,

$AB = A'B'$. Just proved

Also, $CA = C'A'$, and $CB = C'B'$, § 213

(*radii of equal circles*).

$\therefore \triangle ABC \cong \triangle A'B'C'$, § 123

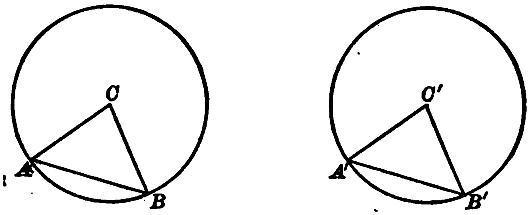
(*having three sides equal*).

$\therefore \angle C = \angle C'$, § 85

(*homologous* $\angle$*s of* $\cong \triangle$*s*). Q. E. D.

Proposition III. Theorem.

234. *In the same circle, or in equal circles, equal chords subtend equal central angles and equal arcs.*



Given $\odot C = \odot C'$, **and chord** $AB =$ **chord** $A'B'$.

To prove $\measuredangle C = \measuredangle C'$, *and arc* $AB =$ *arc* $A'B'$.

Proof. In $\triangle$s ABC and $A'B'C'$,

$CA = C'A'$ and $CB = C'B'$, § 213

(*radii of equal circles*),

and $AB = A'B'$. Hyp.

$\therefore \triangle ABC \cong \triangle A'B'C'$, § 123

(*having three sides equal*).

$\therefore \measuredangle C = \measuredangle C'$, § 85

(*homologous* $\measuredangle$*s of* $\cong \triangle$*s*).

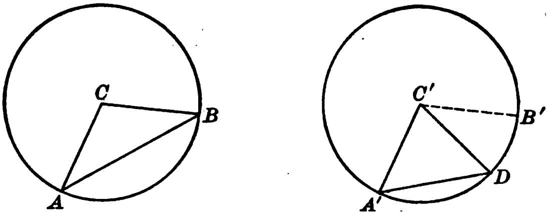
$\therefore$ arc $AB =$ arc $A'B'$. § 230

Q. E. D.

Ex. 101. A circle has only one center.

Proposition IV. Theorem.

235. *In the same circle, or in equal circles, the greater central angle intercepts the greater chord and the greater arc.*



Given $\odot C = \odot C'$, and $\measuredangle ACB > \measuredangle A'C'D$.

To prove chord $AB >$ chord $A'D$, and arc $AB >$ arc $A'D$.

Proof. In $\triangle$s ACB and $A'C'D$,

$CA = C'A'$ and $CB = C'D$, § 213

(*radii of equal circles*),

and $\measuredangle ACB > \measuredangle A'C'D$. Hyp.

$\therefore$ chord $AB >$ chord $A'D$. § 131

Again, draw radius $C'B'$, making $\measuredangle A'C'B' = \measuredangle ACB$. § 125

Then $\measuredangle A'C'B' > \measuredangle A'C'D$.

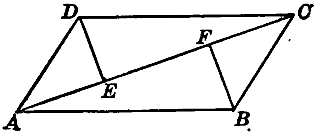
Hence $C'D$ lies between $C'A'$ and $C'B'$, and therefore the point D lies between the points A' and B'.

$\therefore$ arc $A'B' >$ arc $A'D$.

But arc $A'B' =$ arc AB. § 230

$\therefore$ arc $AB >$ arc $A'D$.

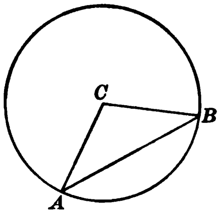
Q. E. D.

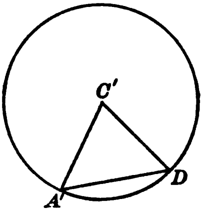


Ex. 102. If perpendiculars, DE and BF, are drawn from the vertices, B and D, of parallelogram $ABCD$ to the diagonal AC, $DE = BF$.

PROPOSITION V. THEOREM.

236. *In the same circle, or in equal circles, the greater arc is intercepted by the greater central angle and subtended by the greater chord.*





Given $\odot C = \odot C'$, **and arc** $AB >$ **arc** $A'D$.

To prove $\angle C > \angle C'$, *and chord* $AB >$ *chord* $A'D$.

Proof. Now, $\angle C = \angle C'$, or $\angle C < \angle C'$, or $\angle C > \angle C'$.

But if $\angle C = \angle C'$, arc $AB =$ arc $A'D$. § 230

And if $\angle C < \angle C'$, arc $AB <$ arc $A'D$, § 235

both of which conclusions are contrary to the hypothesis.

$$\therefore \angle C > \angle C'.$$

Again, in $\triangle$s ACB and $A'C'D$,

$CA = C'A'$ and $CB = C'D$, § 213

(*radii of equal circles*),

and $\angle C > \angle C'$. Just proved

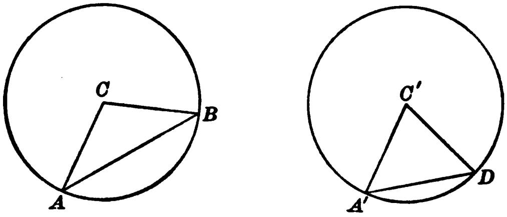
$\therefore$ chord $AB >$ chord $A'D$. § 131

Q. E. D.

Ex. 103. If one leg of a right triangle is half the hypotenuse, the angle opposite that leg is half the other acute angle. (Converse of Ex. 86.)

PROPOSITION VI. THEOREM.

237. *In the same circle, or in equal circles, the greater chord subtends the greater central angle and the greater arc.*



Given $\odot C = \odot C'$, **and the chord** $AB >$ **chord** $A'D$.

To prove $\measuredangle C > \measuredangle C'$, *and arc* $AB >$ *arc* $A'D$.

Proof. In $\triangle$s ACB and $A'C'D$,

$CA = C'A'$ and $CB = C'D$, § 213

(*radii of equal circles*),

and $AB > A'D$. Hyp.

$\therefore \measuredangle C > \measuredangle C'$. § 132

$\therefore$ arc $AB >$ arc $A'D$. § 235

Q. E. D.

Ex. 104. Two circles cannot bisect each other.

Construct a right triangle, having given:

Ex. 105. The two legs.

Ex. 106. The hypotenuse and one leg.

Ex. 107. One leg and the adjacent angle.

Ex. 108. The hypotenuse and an acute angle.

Ex. 109. One leg and the altitude upon the hypotenuse.

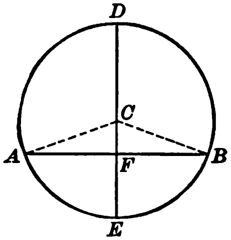
Ex. 110. The hypotenuse and the altitude upon the hypotenuse.

Ex. 111. An acute angle and the sum of the legs.

Ex. 112. An acute angle and the difference of the legs.

PROPOSITION VII. THEOREM.

238. *A diameter perpendicular to a chord bisects the chord and its subtended arcs.*



Given $\odot$ C, with diameter DE $\perp$ chord AB.

To prove *DE bisects chord AB and arcs AEB and BDA.*

Proof. Draw radii CA and CB.

Since $CA = CB$, being radii of the same circle, the $\triangle ACB$ is isosceles.

But $CE \perp AB$. Hyp.

$\therefore CE$ bisects AB and $\measuredangle ACB$, § 101

(*the straight line through the vertex of an isosceles triangle perpendicular to the base bisects the base and the vertical* $\measuredangle$).

$\therefore AF = BF.$

And since $\measuredangle ACF = \measuredangle FCB$, as proved, arc $AE =$ arc EB, § 230

(*in the same circle equal central* $\measuredangle$*s intercept equal arcs*).

Again, $\measuredangle ACD = \measuredangle BCD$, being supplements of equal $\measuredangle$s ACE and BCE. §§ 60, 61

$\therefore$ arc $AD =$ arc BD. § 230

Q. E. D.

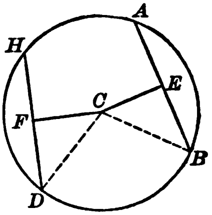
239. COR. 1. *The perpendicular bisector of a chord passes through the center of the circle and bisects the arcs of the chord.*

240. Cor. 2. *Ten propositions, in all, may be obtained by selecting any two of the following conditions for the hypothesis and any one of the remaining three for the conclusion:*

A straight line that
- 1. passes through the center of the circle.
- 2. bisects the chord.
- 3. is perpendicular to the chord.
- 4. bisects the minor arc.
- 5. bisects the major arc.

Proposition VIII. Theorem.

241. *In the same circle, or in equal circles, equal chords are equidistant from the center; and conversely, chords which are equidistant from the center are equal.*



Given, 1, the ⊙ C, with chord AB = chord HD.

To prove *AB and HD are equidistant from the center C.*

Proof. Draw *CE* and *CF* ⊥, respectively, to *AB* and *HD*. § 114

Draw radii *CB* and *CD*.

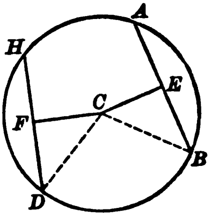
CE and *CF* bisect *AB* and *HD*, respectively, § 240

(*a straight line from the center* ⊥ *a chord bisects the chord*).

Then in rt. △s *CEB* and *CFD*,

$$EB = FD,$$

(*being halves of equal chords*).



$CB = CD,$ § 213

(*being radii of the same circle*).

$\therefore \triangle CEB \cong \triangle CFD,$ § 122

(*two rt. △s are ≅ if the hypotenuse and leg of the one are equal, respectively, to the hypotenuse and leg of the other*).

$\therefore CE = CF,$ § 85

(*homologous sides of ≅ △s*).

Given 2, CE ⊥ AB, CF ⊥ HD, and CE = CF.

To prove $AB = HD.$

Proof. In rt. △s CEB and CFD,

$CE = CF.$ Hyp.

$CB = CD,$ § 213

(*radii of same circle*).

$\therefore \triangle CEB \cong \triangle CFD.$ § 122

$\therefore EB = FD,$ § 85

(*homologous sides of ≅ △s*).

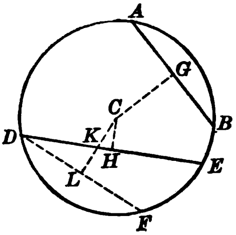
But $2\,EB = AB$, and $2\,FD = HD,$ § 240

(*a line from the center of a circle ⊥ a chord bisects the chord*).

$\therefore AB = HD.$ Q. E. D.

PROPOSITION IX. THEOREM.

242. *In the same circle, or in equal circles, of two unequal chords the greater is nearer the center, and, conversely, the chord nearer the center is the greater.*



Given **I, the ⊙ C with the chord DE > the chord AB, and CH and CG ⊥s from C to DE and AB, respectively.**

To prove $CG > CH$.

Proof. Since the chord $DE >$ the chord AB by hypothesis, arc $DE >$ arc AB. § 237

From D, on the arc DE, let arc DF be taken = arc AB, and draw DF.

Then chord $DF =$ chord AB. § 233

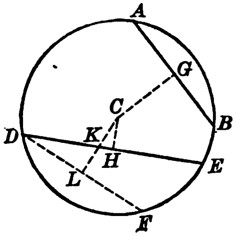
Draw $CL \perp DF$ and intersecting DE at K.

Then $CL = CG$. § 241

But $CL > CK$, Ax. 7

and $CK > CH$. § 116

$\therefore CL$, or its equal CG, $> CH$. Ax. 9



CONVERSELY: Q. E. D.

Given the ⊙ C with chord DE nearer the center than chord AB.

To prove chord $DE >$ chord AB.

Proof. Chord DE is not equal to chord AB, for then would CH equal CG, § 241

(equal chords are equidistant from the center).

Chord DE cannot be less than chord AB, for then would CH be greater than CG. Just proved

$\therefore$ chord $DE >$ chord AB. Q. E. D.

243. COR. *The diameter is greater than any other chord.*

Ex. 113. A chord perpendicular to a diameter through a given point within a circle is the least chord that can be drawn through that point.

Ex. 114. Find the locus of the mid-points of equal chords of a circle.

Ex. 115. Find the locus of the mid-points of parallel chords of a circle.

Ex. 116. If two equal chords intersect, their segments are, respectively, equal.

Ex. 117. If two intersecting chords make equal angles with the diameter through their point of intersection, the chords are equal.

Ex. 118. In a given circle to draw a chord equal and parallel to a given chord.

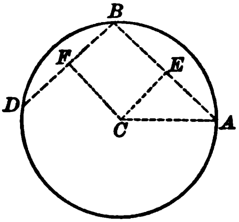
244. DEF. If all the vertices of a polygon lie on a circle, the polygon is said to be **inscribed** in the circle, and the circle said to be **circumscribed** about the polygon.

If all the sides of a polygon are tangents to a circle, the polygon is said to be **circumscribed** about the circle, and the circle to be **inscribed** in the polygon.

245. Def. Points that lie on the same circle are called **concyclic** points. A polygon whose vertices are concyclic is called a **cyclic** polygon.

Proposition X. Theorem.

246. *Through three points not in the same straight line one circle, and only one, can be drawn.*



Given A, B, D, any three points not in a straight line.

To prove one ⊙, and only one, can be drawn through A, B, and D.

Proof. Draw AB and BD.

Construct the ⊥ bisectors of AB and BD. § 113

These ⊥s must meet at some point, C. § 138

C is equidistant from A, B, and D. § 108

Therefore, a circle with C as a center and CA as a radius will pass through A, B, and D.

Since C is the only point of intersection of these ⊥ bisectors, there is no other point equidistant from A, B, and D. § 10 (*b*)

∴ there can be but one circle drawn through A, B, and D.

Q. E. D.

247. Cor. 1. *A circle can be circumscribed about a triangle.*

248. Cor. 2. *Three points, not collinear, determine a circle.*

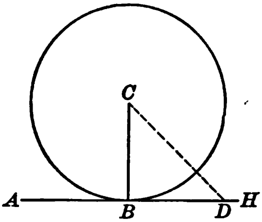
249. Cor. 3. *A circle cannot be drawn through three collinear points.*

250. Cor. 4. *A straight line cannot intersect a circle in more than two points.*

251. Cor. 5. *Two circles cannot intersect in more than two points.*

Proposition XI. Theorem.

252. *A straight line perpendicular to a radius at its extremity is tangent to the circle.*



Given CB, a radius of the ⊙ C, and AH ⊥ CB at its extremity B.

To prove *AH is tangent to the circle C.*

Proof. From the center C draw CD to any other point D in AH.

Since $CB \perp AH$, Hyp.

$CD > CB$. § 116

$\therefore$ the point D lies without the circle.

Hence B is the only point common to the line AH and the circle.

$\therefore$ AH is tangent to the circle C. § 219

Q. E. D.

253. Cor. 1. *A tangent is perpendicular to the radius drawn to the point of contact.*

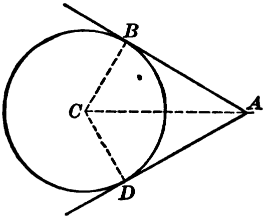
254. Cor. 2. *The perpendicular to a tangent at the point of contact passes through the center of the circle.*

255. Cor. 3. *The line from the center of a circle perpendicular to a tangent passes through the point of contact.*

256. Def. By the tangent from a given point to a given circle is meant that part of the tangent between the given point and the point of contact.

Proposition XII. Theorem.

257. *The tangents to a circle from an external point are equal.*



Given the tangents AB and AD from the point A to the ⊙ C.

To prove $AB = AD$.

Proof. Draw the radii CB and CD, and draw CA.

Then in $\triangle$s ABC and ADC,

$CB = CD$,

(being radii of the same circle).

$CA = CA$, Iden.

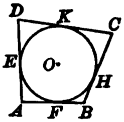
$\measuredangle CBA = \measuredangle CDA$, both being rt. $\measuredangle$s. § 253

$\therefore \triangle ABC \cong \triangle ADC$. § 122

$\therefore AB = AD$, § 85

(homologous sides of $\cong$ $\triangle$s). Q. E. D.

258. Cor. *The tangents from an external point to the same circle make equal angles with the line joining that point and the center of the circle.*



Ex. 119. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

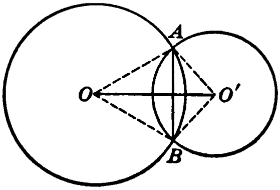
259. Def. The line determined by the centers of two circles is called their **center-line**; the segment of the center-line between the centers of the circles is called the **center-sect**.

260. Def. Two circles having a common center are said to be **concentric**.

261. Def. A tangent to two circles is called a **common internal tangent** if it cuts their center-sect, and a **common external tangent** if it does not cut their center-sect.

Proposition XIII. Theorem.

262. *If two circles intersect, their center-line is the perpendicular bisector of their common chord.*



Given the ⊙s O and O' intersecting at A and B.

To prove *OO' is the ⊥ bisector of AB.*

Proof. Draw the radii OA, OB, $O'A$, and $O'B$.

Then $OA = OB$ and $O'A = O'B$, § 213

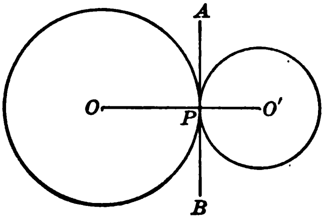
(radii of the same circles).

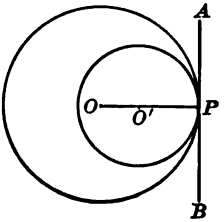
$\therefore OO'$ is the $\perp$ bisector of AB, § 109

(two points equidistant from the extremities of a line determine the $\perp$ bisector of the line). Q. E. D.

PROPOSITION XIV. THEOREM.

263. *If two circles are tangent to each other, the center-line passes through the point of contact.*





Given the $\odot$s O and O′ tangent to each other at the point P.

To prove *P lies on the line OO'.*

Proof. The circles O and O' are tangent to the same straight line AB at P. § 221

But a $\perp$ to AB at P passes through O and O', § 254

(the $\perp$ to a tangent at the point of contact passes through the center of the circle).

$\therefore P$ lies on the line OO', § 52

(only one $\perp$ can be drawn to a given line at a given point in the line). Q. E. D.

Ex. 120. Describe the relative positions of two circles if their center-sect:

(1) is greater than the sum of their radii.

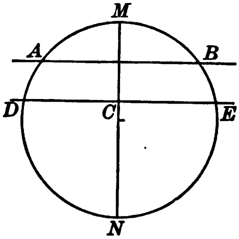
(2) is equal to the sum of their radii.

(3) is less than the sum but greater than the difference of their radii.
(4) is equal to the difference of their radii.
(5) is less than the difference of their radii.
Illustrate by a figure in each case.

Ex. 121. The straight line drawn from the center of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining their points of contact.

PROPOSITION XV. THEOREM.

264. *Parallels intercept equal arcs on a circle.*



Given two ∥s, AB and DE, cutting the ⊙, whose center is C.

To prove arc $AD =$ arc BE.

Proof. Draw the diameter $MN \perp AB$. § 114

Then $MN \perp DE$. § 139

$\therefore$ arc $MD =$ arc ME, § 238

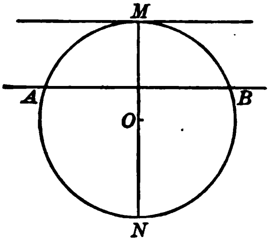
(a diameter ⊥ a chord bisects the chord and its subtended arcs).

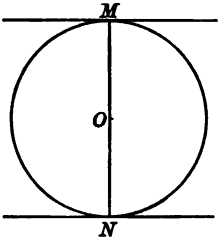
For the same reason arc $MA =$ arc MB.

$\therefore$ arc $MD -$ arc $MA =$ arc $ME -$ arc MB, Ax. 3

or arc $AD =$ arc BE. Q. E. D.

265. Note. The preceding proposition is true however the parallels are drawn; that is, if both are secants, or one is a tangent and the other a secant, or both tangents, and the proof is similar to the foregoing.





MEASUREMENT.

266. The **ratio** of two quantities of the same kind is the quotient obtained by dividing the first quantity by the second.

Thus, if a and b are quantities of the same kind, the ratio of a to b is $\frac{a}{b}$.

267. The **measure** of a quantity is its ratio to another quantity of the same kind, called the **unit of measure.** This ratio is called the **numerical measure** of the quantity.

268. The ratio of two quantities of the same kind is the ratio of their numerical measures.

269. Two quantities are **commensurable** when there is a third quantity, called their **common measure,** that is contained an integral number of times in each.

270. Two quantities are **incommensurable** when they have no common measure.

271. The ratio of two commensurable quantities is called a **commensurable ratio,** and the ratio of two incommensurable quantities is called an **incommensurable ratio.**

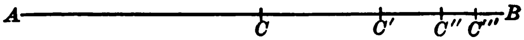
Thus, two lines whose lengths are $8\frac{1}{2}$ and $9\frac{3}{4}$ in., respectively, are commensurable, for they have the common measure $\frac{1}{4}$ in.; but if the lengths of two lines are 2 in. and $\sqrt{3}$ in., respectively, they are incommensurable, for there can be found no line, however small, that will divide each of them without a remainder.

LIMITS.

272. A **constant** is a quantity whose value remains unchanged throughout the same discussion.

273. A **variable** is a quantity which, under a law of change, is subject to a succession of different values in the same discussion.

274. The **limit** of a variable is a constant which the variable, subject to the law of change, continually approaches, but which it cannot exactly reach. That is, the difference between a variable and its limit may be made less than any assignable value, however small, but not zero. A variable may be greater or less than its limit.



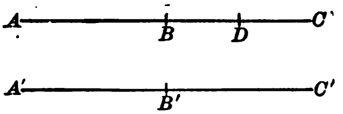
Thus, suppose a point to move from *A* towards *B* under the condition that it shall move the first second half the distance from *A* to *B*, and each succeeding second half the remaining distance to *B*; that is, to the points C, C', C'', C''', etc., respectively. Then it is evident that the moving point may be made to approach *B* as nearly as we please, but that it cannot reach *B*; for, by the condition, it moves each second through half the remaining distance between its position and the point *B*.

Hence the line *AC* is an **increasing variable**, which approaches the line *AB* as its limit, and the line *CB* is a **decreasing variable**, which approaches zero as its limit.

As a second illustration, consider the decimal fraction 333333 . . ., the incomplete reduction of $\frac{1}{3}$ to the decimal form. This may be written $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ etc.

The value of each of these fractions after the first is one tenth that of the preceding one. The series may be continued until the last fraction written differs from zero as little as we please, or less than any assigned value, or until the sum of the series of fractions differs from $\frac{1}{3}$ by less than an assigned value; therefore, the value of the successive fractions approaches zero as a limit, while the value of the sum of all the fractions approaches $\frac{1}{3}$ as a limit.

275. Theorem. *If two variables are constantly equal and each approaches a limit, their limits are equal.*



Given AB and A'B', two equal variables, having the limits AC and A'C', respectively.

To prove $AC = A'C'$.

Proof. Suppose either, as $A'C'$, to be less than the other, as AC. Then on AC suppose AD is taken $= A'C'$.

Then the variable AB, which may be made to differ from AC by less than any assigned quantity, may be made greater than AD, while $A'B'$ is always less than $A'C'$ or its equal AD, and the two variables are, therefore, unequal, which is contrary to the hypothesis.

$$\therefore AC = A'C'.$$ Q. E. D.

276. Theorem. *If the limit of a variable is zero, the limit of the product of the variable by a constant is zero.*

Given a constant k and a variable x, whose limit is zero.

To prove *the limit of kx is zero.*

Proof. Let there be taken any quantity, q, however small but greater than zero.

Then we may choose a value for x so that

$$x < \frac{q}{k}.$$

Whence $kx < q$.

Hence kx may be made less than any assigned quantity, but not zero.

$\therefore$ the limit of kx is zero. Q. E. D.

277. THEOREM. *The limit of the product of a constant and a variable is the product of the constant by the limit of the variable.*

Given a constant k, and a variable x, whose limit is a.

To prove *the limit of kx is ka.*

Proof. Let d be the variable difference between the variable x and its limit a.

Then $x = a + d$.

Hence $kx = ka + kd$.

But the limit of d is zero. § 274

$\therefore$ the limit of kd is zero. § 276

$\therefore$ the limit of kx is ka. § 274

Q. E. D.

278. COR. *The limit of the quotient of a variable divided by a constant is equal to the limit of the variable divided by the constant.*

279. THEOREM. *The limit of the product of two variables is the product of their limits.*

Given the variables x and y, and their respective limits a and b.

To prove *the limit of xy is ab.*

Proof. Let d and d' be the variable differences between x and y and their respective limits.

Then $$x = a + d,$$

and $$y = b + d'.$$

Whence $$xy = ab + (ad' + bd + dd').$$

$$\therefore xy = ab + \text{the terms whose limit is zero.}$$

$\therefore$ the limit of xy is ab. § 274 Q.E.D.

280. Cor. *The limit of the nth power of a variable is the nth power of its limit.*

281. Theorem. *The limit of the sum of two or more variables is the sum of their limits.*

Given x and y, variables, and a and b, their respective limits.

To prove *the limit of* $x + y$ is $a + b$.

Proof: Let d and d' be the variable differences between x and y and their respective limits.

Then $$x = a + d,$$

and $$y = b + d'.$$

Whence $$x + y = a + b + (d + d').$$

But the limit of $d + d'$ is zero. § 274

$\therefore$ the limit of $x + y$ is $a + b$. Q.E.D.

Similarly for three or more variables.

282. Theorem. *If two variables have a constant ratio, and each approaches a limit, the limits have the same ratio.*

Given the variables x and y, having a and b for their respective limits, and the constant ratio $\frac{x}{y} = r$.

To prove $$\frac{a}{b} = \frac{x}{y}.$$

Proof. $\frac{x}{y} = r.$ Hyp.

Hence $x = ry.$

Then the limit of $x =$ the limit of $ry.$ § 275

$= r \times$ the limit of $y,$ § 277

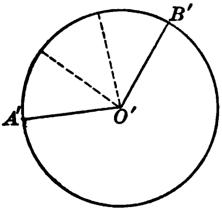
or $a = rb.$

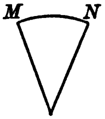
$\therefore \frac{a}{b} = r = \frac{x}{y}.$ Q. E. D.

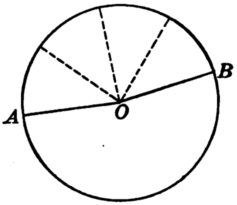
NOTE. In assuming $x = a + d$ in the foregoing demonstrations, x was considered a decreasing variable. The same proofs would hold were x taken as an increasing variable.

PROPOSITION XVI. THEOREM.

283. *In the same circle, or in equal circles, central angles have the same ratio as their intercepted arcs.*







Given the $\odot$s O and O', in which the central $\measuredangle$s AOB and A'O'B' intercept the arcs AB and A'B', respectively.

To prove $\frac{\measuredangle A'O'B'}{\measuredangle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}.$

Proof. CASE 1. When the arcs are commensurable.

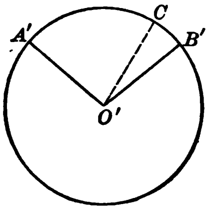
Let the arc MN be a common measure of arcs AB and $A'B'$.

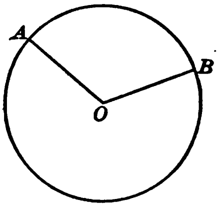
Suppose it is contained in the arc $A'B'$ m times and in the arc AB n times.

Then $\frac{\text{arc } A'B'}{\text{arc } AB} = \frac{m}{n}.$ § 268

If radii be drawn to the points of division, they will divide the $\measuredangle A'O'B'$ into m equal angles and the $\measuredangle AOB$ into n equal angles. § 233

Hence $$\frac{\measuredangle A'O'B'}{\measuredangle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}.$$





Case II. When the arcs are incommensurable.

Suppose the arc AB to be divided into any number of equal parts.

Take one of these parts and let it be applied as a unit of measure to the arc $A'B'$. The arc $A'B'$ will contain this unit a certain number of times and leave a remainder less than the unit.

Suppose the unit of measure to be contained an integral number of times in the arc $A'C$, with CB' as the remainder.

Draw the radius $O'C$.

Then, since the arcs AB and $A'C$ are commensurable,

$$\frac{\measuredangle A'O'C}{\measuredangle AOB} = \frac{\text{arc } A'C}{\text{arc } AB}. \qquad \text{Case I}$$

Let the unit of measure be indefinitely decreased. Then the remainder CB', always less than the unit of measure, must likewise decrease indefinitely.

Hence the arc $A'C$ approaches the arc $A'B'$ as a limit, and $\measuredangle A'O'C$ approaches $\measuredangle A'O'B'$ as a limit.

$$\therefore \frac{\text{arc } A'C}{\text{arc } AB} \text{ approaches } \frac{\text{arc } A'B'}{\text{arc } AB} \text{ as a limit,} \qquad \S\ 278$$

and $$\frac{\measuredangle A'O'C}{\measuredangle AOB} \text{ approaches } \frac{\measuredangle A'O'B'}{\measuredangle AOB} \text{ as a limit.} \qquad \S\ 278$$

But the variables $\frac{\text{arc } A'C}{\text{arc } AB}$ and $\frac{\measuredangle A'O'C}{\measuredangle AOB}$ are always equal.

$$\therefore \frac{\measuredangle A'O'B'}{\measuredangle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}. \qquad \text{§ 275}$$

Q. E. D.

284. Cor. 1. *The ratio of a central angle to a perigon is equal to the ratio of the intercepted arc to the circle.*

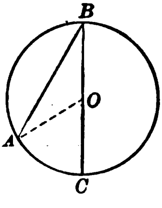
285. Cor. 2. *A straight central angle intercepts a semicircle, and a right central angle intercepts a quadrant.*

286. Cor. 3. *In equal circles two sectors have the same ratio as their central angles.*

287. A circle is supposed to be divided into 360°, hence a unit central angle intercepts a unit arc. Therefore the numerical measure of a central angle is equal to the numerical measure of the intercepted arc. For the sake of brevity and for convenience in comparing the magnitude of angles by means of their intercepted arcs, this relation is expressed by saying, *A central angle is measured by its intercepted arc.*

PROPOSITION XVII. THEOREM.

288. *An inscribed angle is measured by half its intercepted arc.*



Given the $\measuredangle$ ABC inscribed in the $\odot$ O.

To prove $\measuredangle ABC$ is measured by half the arc AC.

Proof. Case I. When one side of the angle is a diameter of the circle.

Draw OA.

Then in $\triangle AOB$, $OA = OB$, being radii of the same circle.

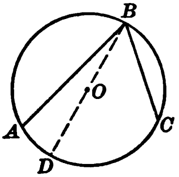
$\therefore \measuredangle ABC = \measuredangle BAO$. § 98

But $\measuredangle AOC = \measuredangle ABC + \measuredangle BAO = 2 \measuredangle ABC$. § 158

$\therefore \measuredangle ABC = \frac{1}{2} \measuredangle AOC$.

But $\measuredangle AOC$ is measured by arc AC. § 287

$\therefore \measuredangle ABC$ is measured by half the arc AC.



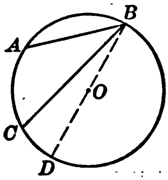
Case II. When the center is within the angle.

Draw the diameter BD.

Then $\measuredangle ABD$ is measured by $\frac{1}{2}$ arc AD, and DBC by $\frac{1}{2}$ arc DC. Case I.

$\therefore \measuredangle (ABD + DBC)$ is measured by $\frac{1}{2}$ arc $(AD + DC)$,

or $\measuredangle ABC$ by $\frac{1}{2}$ arc AC.



Case III. When the center is without the angle.

Draw the diameter BD.

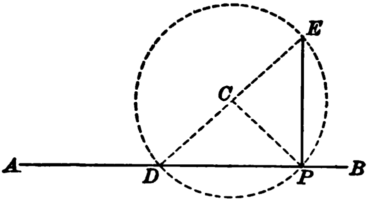
Then $\measuredangle ABD$ is measured by $\frac{1}{2}$ arc AD, and $\measuredangle CBD$ by $\frac{1}{2}$ arc CD. Case I

$\therefore \measuredangle (ABD - CBD)$ is measured by $\frac{1}{2}$ arc $(AD - CD)$,

or $\measuredangle ABC$ by $\frac{1}{2}$ arc AC. Q. E. D.

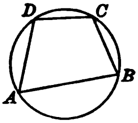
289. Cor. 1. *Angles inscribed in the same segment, or in equal segments, are equal.*

290. Cor. 2. *An inscribed angle is acute, right, or obtuse, according as the segment is greater than, equal to, or less than, a semicircle.*



Note. § 290 suggests another method of drawing a perpendicular to a given line at a given point in the line (see § 113). Given *P*, the point in *AB*. Take as a center any point *C*, without *AB* and not in the perpendicular to *AB* at *P*, and with radius *CP* describe an arc cutting *AB* at *D* and *P*. Draw *DC* and produce it to meet the arc again at *E*. Then *PE* is the perpendicular required, the angle *DPE* being inscribed in a semicircle, and, therefore, a right angle.

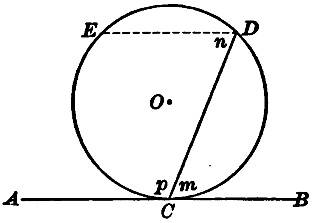
291. Cor. 3. *The circle described upon the hypotenuse of a right triangle as diameter passes through the vertex of the right angle.*



Ex. 122. The opposite angles of an inscribed quadrilateral are supplementary.

Proposition XVIII. Theorem.

292. *The angle formed by a tangent and a chord drawn to the point of contact is measured by half the intercepted arc.*



Given the ⊙ O, and the $\angle$ m, formed by the tangent AB and the chord CD.

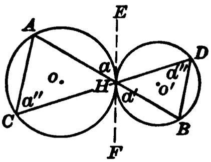
To prove

$\angle m$ *is measured by* $\frac{1}{2}$ *arc CD, and* $\angle p$ *by* $\frac{1}{2}$ *arc DEC.*

Proof.	Draw the chord $DE \parallel AB$.	§ 154
Then	$\angle m = \angle n$.	§ 143
But	$\angle n$ is measured by $\frac{1}{2}$ arc EC.	§ 288
	$\therefore \angle m$ is measured by $\frac{1}{2}$ arc EC.	Ax. 1
But	arc EC = arc CD, (‖s *intercept equal arcs*).	§ 264
	$\therefore \angle m$ is measured by $\frac{1}{2}$ arc CD.	

Also $\angle p$, the supplement of $\angle m$, is measured by $\frac{1}{2}$ arc DEC, the conjugate of arc CD. Q. E. D.

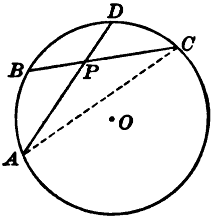
Ex. 123. Using § 292, prove § 257.



Ex. 124. If two circles are tangent to each other, and two secants are drawn through the point of contact, the chords joining their extremities are parallel.

PROPOSITION XIX. THEOREM.

293. *The angle formed by two chords intersecting within the circle is measured by half the sum of the intercepted arcs.*



Given the ⊙ O, and two chords, AD and BC, intersecting at P.

To prove ∡ CPD is measured by $\frac{1}{2}$ (arc CD + arc AB).

Proof. Draw AC.

Then in △ APC, ∡ DPC = ∡ A + ∡ C, § 158

(an exterior ∡ of a △ = the sum of the two non-adjacent int. ∡s).

But ∡ A is measured by $\frac{1}{2}$ arc CD, § 288

(an inscribed ∡ is measured by $\frac{1}{2}$ its intercepted arc).

and ∡ C is measured by $\frac{1}{2}$ arc AB. § 288

∴ ∡ CPD is measured by $\frac{1}{2}$ (arc CD + arc AB). Q. E. D.

Ex. 125. If through any point within a circle two perpendicular chords are drawn, the sum of two opposite intercepted arcs is equal to a semicircle.

PROPOSITION XX. THEOREM.

294. *The angle formed by two tangents, or by a tangent and a secant, or by two secants, intersectiny without the circle, is measured by half the difference of the intercepted arcs.*

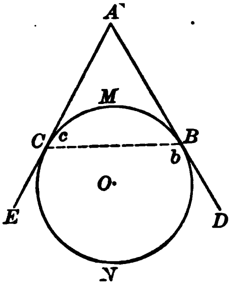


FIG. 1.

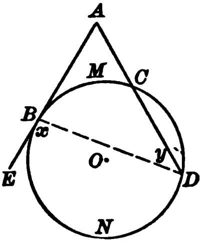


FIG. 2.

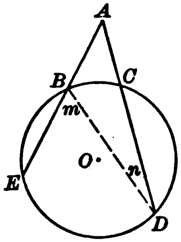


FIG. 3.

I. Given the ⊙ O (Fig. 1), and the two tangents, AD and AE, touching the circle at B and C, respectively.

To prove $\measuredangle A$ *is measured by* $\frac{1}{2}$ *(arc BNC − arc BMC).*

Proof. Draw BC.

Then in $\triangle ABC$, $\measuredangle A + \measuredangle c = \measuredangle b$, § 158

or $\measuredangle A = \measuredangle b - \measuredangle c$.

But $\measuredangle b$ is measured by $\frac{1}{2}$ arc BNC, § 292

and $\measuredangle c$ is measured by $\frac{1}{2}$ arc BMC. § 292

$\therefore \measuredangle A$ is measured by $\frac{1}{2}$ (arc BNC − arc BMC).

II. Given ⊙ O (Fig. 2), and tangent AE and secant AD.

To prove $\measuredangle A$ *is measured by* $\frac{1}{2}$ *(arc BND − arc BMC).*

Proof. Draw BD.

Then $\measuredangle A = \measuredangle x - \measuredangle y$. § 158

But $\measuredangle x$ is measured by $\frac{1}{2}$ arc BND, § 292

and $\measuredangle y$ is measured by $\frac{1}{2}$ arc BMC. § 288

$\therefore \measuredangle A$ is measured by $\frac{1}{2}$ (arc BND − arc BMC).

III. Given ⊙ O (Fig. 3) and two secants AE and AD.

To prove $\measuredangle A$ *is measured by* $\frac{1}{2}$ *(arc DE − arc BC).*

Proof. Draw BD.

Then $\measuredangle A = \measuredangle m - \measuredangle n$. § 158

But $\measuredangle m$ is measured by $\frac{1}{2}$ arc DE, § 288

and $\measuredangle n$ is measured by $\frac{1}{2}$ arc BC. § 288

$\therefore \measuredangle A$ is measured by $\frac{1}{2}$ (arc DE − arc BC). Q.E.D.

CORRELATIVITY AND CONTINUITY.

295. Some pairs of geometric concepts differ in meaning only in that they denote opposition in direction, position, operation, etc. The names of such concepts are called **correlative terms**, and two theorems, formed each from the other, by the interchange of correlative terms, are called **correlative theorems.**

Correlative Terms.

interior,	exterior.
internal,	external.
inscribed,	circumscribed.
acute,	obtuse.
sum,	difference.
positive,	negative.
bisector of angle,	perpendicular bisector of side.
maximum,	minimum.
greater,	less.

296. There is sometimes an *intermediate* term corresponding to two correlative terms; as **right** is the intermediate between *acute* and *obtuse;* **equal** the intermediate between *greater* and *less*, and the interchanging of such terms gives a series of three kindred theorems, susceptible of being stated in general terms as one theorem. Thus all the theorems concerning the measurement of angles by arcs of a circle, the sides of the angle intersecting *within*, **on**, or *without* the circle, may be included in the general theorem:

The angle formed by two intersecting lines cutting or touching a circle is measured by half the **algebraic** *sum of the intercepted arcs; both arcs being considered* **positive** *when the lines intersect* **within** *the circle; one arc as* **zero** *when they intersect* **on** *the circle; and the greater arc positive and the lesser arc* **negative** *when they intersect* **without** *the circle.*

This method of treatment applied to the different relative positions of lines, etc., is called the **Principle of Continuity.**

Ex. 126. The bisectors of the angles of a circumscribed polygon pass through the center of the circle.

Ex. 127. The perpendicular bisectors of the sides of an inscribed polygon pass through the center of the circle. (Correlative of Ex. 126.)

Ex. 128. The shorter segment of the diameter through a given point within a circle is the shortest line that can be drawn from that point to the circle.

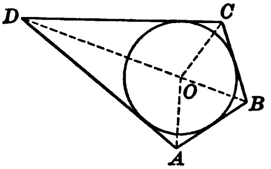
Ex. 129. The longer segment of the diameter through a given point within a circle is the longest line that can be drawn from that point to the circle. (Correlative of Ex. 128.)

Ex. 130. If two opposite angles of a quadrilateral are supplementary, a circle may be circumscribed about the quadrilateral. (Converse of Ex. 122.)

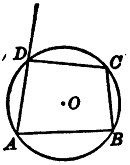
Ex. 131. The sum of the legs of a right triangle is equal to the sum of the hypotenuse and the diameter of the inscribed circle.

Ex. 132. If two circles are tangent externally to each other, the tangents to them from any point of their common interior tangent are equal.

Ex. 133. If the angle formed by two tangents is 80°, how many degrees are in each of the intercepted arcs?

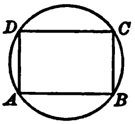


Ex. 134. The central angles subtended by any two opposite sides of a circumscribed quadrilateral are supplementary.

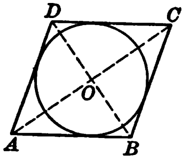


Ex. 135. An exterior angle of an inscribed quadrilateral is equal to the opposite interior angle.

Ex. 136. Two chords perpendicular to a third chord at its extremities are equal.



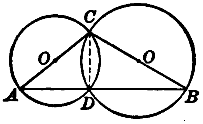
Ex. 137. An inscribed parallelogram is a rectangle.



Ex. 138. A circumscribed parallelogram is a rhombus or a square.

Ex. 139. All chords of a circle tangent to an interior concentric circle are equal and are bisected at the points of contact.

Ex. 140. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.



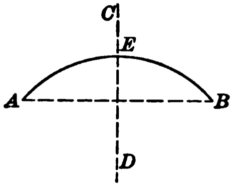
Ex. 141. Circles drawn on two sides of a triangle as diameters intersect on the third side.

Ex. 142. If the angle formed by a tangent and a secant is 20° and the greater intercepted arc is 80°, what is the lesser intercepted arc?

Ex. 143. The circle described upon one of the legs of an isosceles triangle as diameter bisects the base.

PROPOSITION XXI. PROBLEM.

297. *To bisect a given arc.*



Given the arc AB.

Required to bisect the arc AB.

Construction. Draw the chord *AB*.

Draw *CD*, the ⊥ bisector of chord *AB*. § 112

Then *CD* bisects the arc *AB* at *E*.

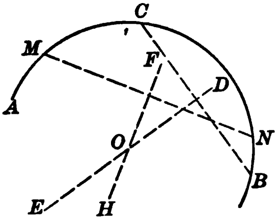
Proof. *CD* is the ⊥ bisector of the chord *AB*. Cons.

∴ *CD* bisects arc *AB*, § 239

(*the ⊥ bisector of a chord passes through the center of the circle and bisects the arcs of the chord*). Q. E. F.

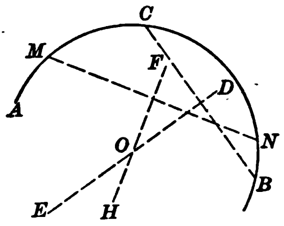
PROPOSITION XXII. PROBLEM.

298. *To find the center of a given arc.*



Given the arc AB.

Required to find the center of the arc AB.



Construction. Draw any two chords of the arc, as *MN* and *BC*.

Draw *DE* and *FH*, the ⊥ bisectors of chords *MN* and *BC*. § 112

These ⊥s meet in some point *O*. § 138

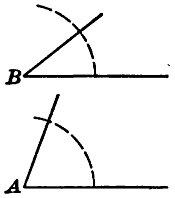
But the center lies on *DE* and *FH*. § 239

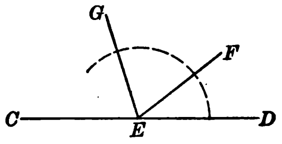
∴ the point *O* is the center required. § 10 (*b*)

Q. E. F.

Proposition XXIII. Problem.

299. *Given two angles of a triangle, to construct the third angle.*





Given ∡s A and B, two angles of a △.

Required to construct the third angle of the △.

Construction. Draw any st. line *CD* and at any point *E* on the line construct ∡ *DEF* = ∡ *B*, and ∡ *FEG* = ∡ *A*. § 125

Then ∡ *GEC* is the angle required.

Proof. Since the sum of the three angles of a $\triangle =$ a st. $\measuredangle$, § 155

and $\measuredangle DEF + \measuredangle FEG + \measuredangle GEC =$ a st. $\measuredangle$, § 62

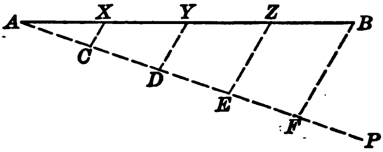
then the sum of the $\measuredangle$s DEF, FEG, and $GEC =$ the sum of the $\measuredangle$s of the $\triangle$.

But $\measuredangle DEF = \measuredangle B$ and $\measuredangle FEG = \measuredangle A$. Cons.

$\therefore$ the third $\measuredangle$ of the $\triangle = \measuredangle GEC$. Q. E. F.

PROPOSITION XXIV. PROBLEM.

300. *To divide a given straight line into any given number of equal parts.*



Given the st. line AB.

Required to divide AB into any given number of equal parts.

Construction. Draw any indefinite line AP.

Take AC, a line of convenient length, and apply it to AP as many times as AB is to be divided into equal parts.

From F, the last point of division, draw FB.

Through E, D, and C, the other points of division, draw $\parallel$s to FB, § 154, cutting AB in X, Y, and Z. These lines divide AB as required.

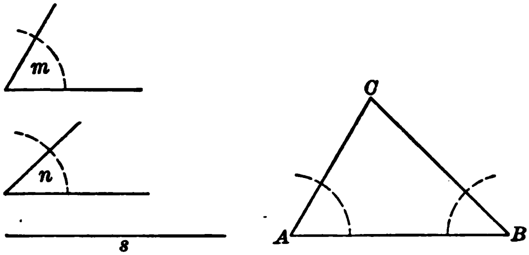
Proof. The $\parallel$s CX, DY, and FB intercept equal parts of AP. Cons.

$\therefore$ AB is divided into equal parts, § 182

(if a series of $\parallel$s intercept equal lengths on one transversal, they intercept equal lengths on every transversal). Q. E. F.

PROPOSITION XXV. PROBLEM.

301. *To construct a triangle, having given two angles and the included side.*



Given ∡s m and n, and the included side s.

Required to construct the $\triangle$.

Construction. Draw the line $AB = s$.

At A construct an $\measuredangle = \measuredangle m$ § 125

At B construct an $\measuredangle = \measuredangle n$. § 125

Produce the sides of these ∡*s* until they meet at C.

Then $\triangle ABC$ is the $\triangle$ required.

Proof. Since $\measuredangle A = \measuredangle m$, $\measuredangle B = \measuredangle n$, and $AB = s$. Cons.

$\therefore \triangle ABC$ is the $\triangle$ required. § 94

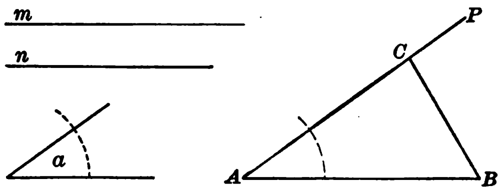
Q. E. F.

Discussion. The problem is impossible if the sum of the two given angles is not less than a straight angle. § 155

Ex. 144. To construct a triangle, having given two angles and the side opposite one of them. (Find the third angle by 299 and apply 301.)

PROPOSITION XXVI. PROBLEM.

302. *To construct a triangle, having given two sides and the included angle.*



Given m and n, two sides of a △, and a, the included $\measuredangle$.

Required to construct the △.

Construction. Draw $AB = m$.

At A construct $\measuredangle BAP = \measuredangle a$. § 125

On AP lay off $AC = n$.

Draw BC.

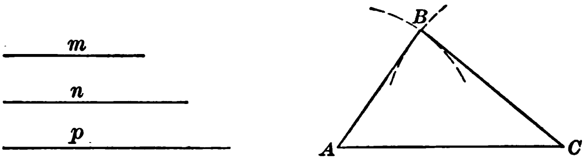
Then △ ABC is the △ required.

Proof. △ ABC has by construction two sides and the included angle = the given sides and included angle. § 91

Q. E. F

Proposition XXVII. Problem.

303. *To construct a triangle, having given the three sides.*



Given m, n, and p, the sides of a △.

Required to construct the △.

Construction. Draw $AC = p$.

With A and C as centers and radii = m and n, respectively, describe arcs intersecting at B.

Draw BA and BC.

Then △ ABC is the △ required.

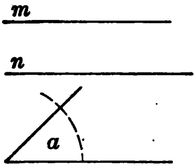
Proof. The $\triangle ABC$ has its sides equal by construction to the given sides. § 123

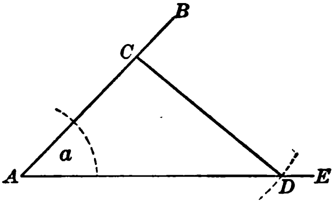
Q.E.F.

Discussion. The problem is impossible when one of the given sides is not less than the sum of the other two. § 86

PROPOSITION XXVIII. PROBLEM.

304. *To construct a triangle, having given two sides and the angle opposite one of them.*





Given m and n, two sides of a $\triangle$, and $\measuredangle$ a, opposite the side n.

Required to construct the $\triangle$.

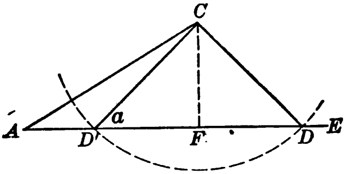
Construction.

CASE I. When $\measuredangle a$ is acute.

Construct $\measuredangle BAE = \measuredangle a$ (§ 125), and on AB lay off $AC = m$.

With C as a center and a radius n, describe an arc cutting AE at D. Draw CD.

Then $\triangle ACD$ is the $\triangle$ required, having the two sides and the angle equal to the given parts. Q.E.F.

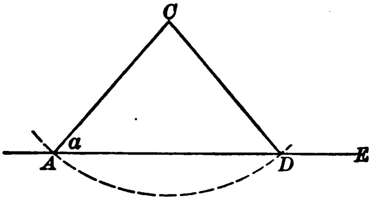


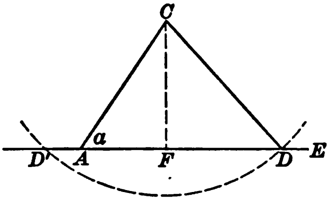
Discussion. 1. When $n < CF$, the perpendicular from C to AE, no solution is possible.

2. When $n = CF$, the arc will touch AE, and there is one solution, the rt. $\triangle\, ACF$.

3. When $n > CF$ but $< m$, the arc cuts AE in two points, D and D', making $FD = FD'$ (§ 118), and there are two solutions, the $\triangle$s ACD and ACD'.

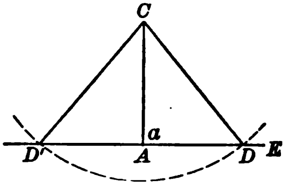
This is called **the ambiguous case.**





4. When $n = m$, the arc cuts AE in D and A, and there is one solution, the isosceles $\triangle\, ACD$.

5. When $n > m$, the arc cuts AE in D and D', making $FD' = FD > FA$ (§ 120); therefore A lies between D and D', and there is one solution, the $\triangle\, ACD$.

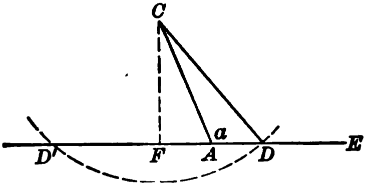


CASE II. When $\measuredangle\, a$ is a rt. $\measuredangle$.

Then $$n > m. \qquad \text{§ 116}$$

$$AD = AD'. \qquad \text{§ 118}$$

Hence there is one solution, the congruent rt. $\triangle$s ACD and ACD'.



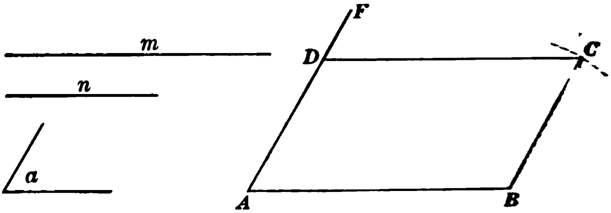
Case III. When ∡ a is obtuse.

Then $n > m$, § 126

and F lies between A and D', and there is one solution, the obtuse △ ACD.

Proposition XXIX. Problem.

305. *To construct a parallelogram, having given two sides and the included angle.*



Given two sides, m and n, and the included ∡ a.

Required to construct the ▱.

Construction. Draw $AB = m$.

Construct ∡ FAB = ∡ a. § 125

On AF take $AD = n$.

With B and D as centers, and with radii n and m, respectively, describe arcs intersecting at C.

Draw BC and DC.

Then $ABCD$ is the ▱ required.

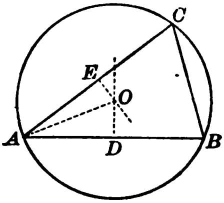
Proof. $AD = BC$, and $AB = DC$. Cons.

∴ $ABCD$ is a ▱, § 172

(If the opposite sides of a quadrilateral are equal, it is a ▱). Q.E.F.

PROPOSITION XXX. PROBLEM.

306. *To circumscribe a circle about a given triangle.*



Given the △ ABC.

Required to circumscribe a ⊙ about the △ ABC.

Construction. Construct the ⊥ bisectors of any two of the sides, as AB and AC, at D and E, respectively. § 112

These ⊥s will meet in some point, O. § 138

Then with O as a center and a radius OA describe a ⊙.

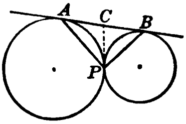
This will be the ⊙ required.

Proof. $OA = OB = OC$. § 200

∴ O is the center of the ⊙ through A, B, and C. Q.E.F.

Ex. 145. To construct a triangle, having given two sides and the radius of the circumscribed circle.

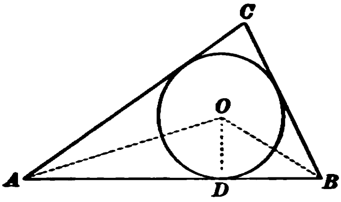
Ex. 146. The line through the mid-points of two parallel chords passes through the center of the circle.



Ex. 147. Two circles are tangent at P, and a common exterior tangent touches them at A and B, respectively. Prove the angle APB is a right angle.

PROPOSITION XXXI. PROBLEM.

307. *To inscribe a circle in a given triangle.*



Given the △ ABC.

Required to inscribe a ⊙ in △ ABC.

Construction. Bisect any two ∡s of the △, as ∡s A and B. § 129

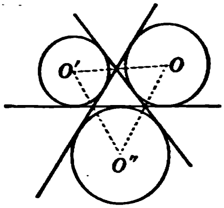
The bisectors will meet in some point, O. § 196

Draw $OD \perp AB$. § 114

Then with O as a center and OD as radius describe a ⊙. This will be the ⊙ required.

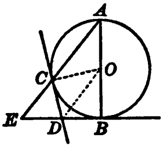
Proof. O is equidistant from AB, AC, and BC. § 196

∴ The ⊙ whose center is O and radius OD will touch each side of the △. Q.E.F.



308. NOTE. An escribed circle is tangent to one side of a triangle and to the other two sides produced. The center of an escribed circle is the intersection of the bisectors of the exterior angles of a triangle. A triangle has three escribed circles. The centers of the escribed circles are called the **ex-centers** of the triangle.

Ex. 148. If a circle is described upon one leg of a right triangle as diameter, the tangent through the point of intersection of the circle and the hypotenuse bisects the other leg.



PROPOSITION XXXII. PROBLEM.

309. *To draw through a given point a tangent to a given circle.*

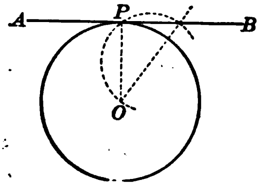


FIG. 1.

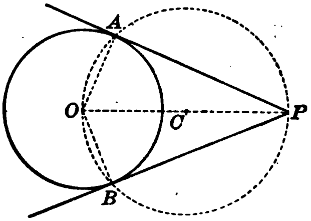


FIG. 2.

CASE I. When the given point is on the circle.

Given any point P, on the ⊙ whose center is O (Fig. 1).

Required to draw through P a tangent to the ⊙ O.

Construction. Draw *PO*, and through *P* draw *AB* ⊥ *OP*. § 290

Then *AB* is the tangent required.

Proof. *AB* is ⊥ *OP*. Cons.

∴ *AB* is tangent to ⊙ *O*, § 252

(a st. line ⊥ a radius at its extremity is tangent to the ⊙).

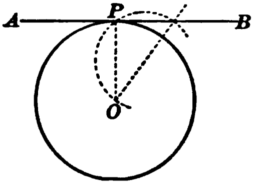


Fig. 1.

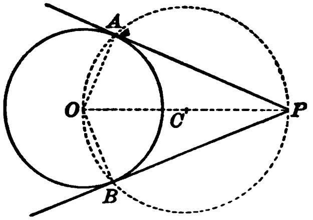


Fig. 2.

Case II. When the given point is without the given ⊙.

Given any point P, without the ⊙ O (Fig. 2).

Required to draw through P a tangent to the ⊙ O.

Construction. Draw OP and bisect OP at C. § 112

With C as a center and CP as a radius describe a ⊙, intersecting the given circle at A and B.

Draw PA and PB.

Then PA or PB is the tangent required.

Proof. Draw OA and OB.

∡s OAP and OBP are rt. ∡s, § 290

(being inscribed in a semicircle).

∴ PA and PB are tangents to the ⊙ O. § 252

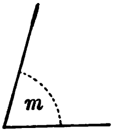
Q. E. F.

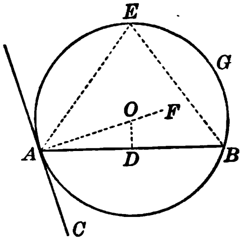
Ex. 149. To construct a triangle, having given one side, an adjacent angle, and the radius of the inscribed circle.

Ex. 150. Construct a right triangle having given one leg and the radius of the circumscribed circle.

Proposition XXXIII. Problem.

310. *Upon a given straight line as a chord, to construct a segment of a circle which shall contain a given angle.*





Given the line AB, and the $\measuredangle$ m.

Required to construct upon AB as a chord, a segment of a $\odot$ in which any inscribed $\measuredangle = \measuredangle m$.

Construction. Construct $\measuredangle BAC = \measuredangle m$. § 125

Draw $AF \perp AC$. § 113

Draw DO, the $\perp$ bisector of AB, intersecting AF at O. § 112

With O as a center, and OA as a radius, describe the $\odot ABG$.

Then AEB is the required segment.

Proof. $OA = OB$. § 108

$\therefore$ the $\odot$ passes through B, and AC is a tangent. § 252

Then $\measuredangle BAC$ is measured by $\frac{1}{2}$ arc AB. § 292

But the inscribed $\measuredangle AEB$ is measured by $\frac{1}{2}$ arc AB. § 288

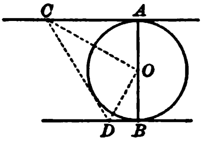
$$\therefore \measuredangle AEB = \measuredangle BAC = \measuredangle m.$$

$\therefore$ the segment AEB contains the required $\measuredangle$. Q.E.F.

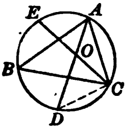
Ex. 151. If a circle is described upon the radius of another circle as diameter, any chord of the greater circle through the point of tangency is bisected by the smaller.

Ex. 152. The angle formed by two tangents from a point whose distance from the center of the circle is equal to the diameter, is one third a straight angle.

Ex. 153. If tangents CA and DB are drawn to the extremities of the diameter AB to meet a third tangent at C and D, the angle COD is a right angle.



Ex. 154. The bisectors of the vertical angles of all triangles having equal vertical angles and standing on the same side of the same base are concurrent.



Ex. 155. O is the in-center of an inscribed triangle ABC. AO is produced to meet the circle at D. Prove that $OD = DC$.

Ex. 156. Construct the complement of a given angle.

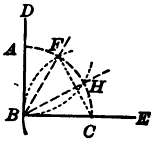
Ex. 157. Construct the supplement of a given angle.

Ex. 158. Given three angles of a quadrilateral, to construct the fourth angle.

Ex. 159. Construct an angle of 45°.

Ex. 160. Construct an angle of 60°.

Ex. 161. Construct an angle of 75°.



Ex. 162. Trisect a right angle.

Ex. 163. Trisect a straight angle.

Ex. 164. Draw a tangent to a given circle so that it shall be parallel to a given line.

Ex. 165. Draw a tangent to a given circle so that it shall be perpendicular to a given line.

Ex. 166. Draw a tangent to a given circle so that it shall make with a given line an angle equal to a given angle.

Construct an equilateral triangle, having given:

Ex. 167. One side.

Ex. 168. The altitude.

Ex. 169. The radius of the inscribed circle.

Ex. 170. The perimeter.

Construct an isosceles triangle, having given:

Ex. 171. The base and the radius of the circumscribed circle.

Ex. 172. The base and the radius of the inscribed circle.

Ex. 173. The altitude and an angle at the base.

Ex. 174. The altitude and the vertical angle.

Construct a triangle, having given:

Ex. 175. One side, an adjacent angle, and the sum of the other two sides.

Ex. 176. One side, an adjacent angle, and the difference of the other two sides.

Ex. 177. Two angles and the radius of the circumscribed circle.

Ex. 178. Two angles and the radius of the inscribed circle.

Ex. 179. Two sides and the median to the third side.

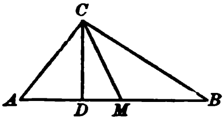
Ex. 180. One side and the medians to the other sides.

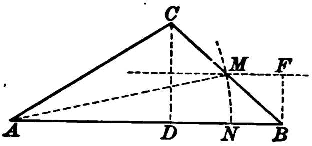
Ex. 181. The three medians.

Ex. 182. Two sides and the altitude to the third side.

Ex. 183. One side and the altitudes to the other sides.

Ex. 184. One side and the median and the altitude to that side.





Ex. 185. One side and the median and the altitude to another side.

Ex. 186. One side, the altitude to a second side, and the median to the third side.

Ex. 187. One side, an adjacent angle, and the radius of the circumscribed circle.

Ex. 188. The mid-points of the sides.

Ex. 189. The base, the altitude, and the vertical angle.

Ex. 190. The base, the corresponding median, and the vertical angle.

Construct a square, having given:

Ex. 191. The side.

Ex. 192. The sum of diagonal and side.

Ex. 193. The difference between diagonal and side.

Construct a rectangle, having given:

Ex. 194. Two adjacent sides.

Ex. 195. One side and the diagonal.

Ex. 196. The perimeter and the diagonal.

Ex. 197. The perimeter and the angle formed by the diagonals.

Ex. 198. One side and the angle formed by the diagonals.

Construct a rhombus, having given:

Ex. 199. One side and one angle.

Ex. 200. One side and one diagonal.

Ex. 201. One angle and one diagonal.

Ex. 202. One side and the radius of the inscribed circle.

Construct a parallelogram, having given:

Ex. 203. Two sides and a diagonal.

Ex. 204. The diagonals and the angle between them.

Ex. 205. One side and the two diagonals.

Ex. 206. One side, one angle, and one diagonal.

Construct a trapezoid, having given:

Ex. 207. The bases and the legs.

Ex. 208. The bases and the diagonals.

Ex. 209. The bases, one diagonal, and the angle between the diagonals.

Ex. 210. One base, one angle at this base, and the two legs.

Find the locus of:

Ex. 211. The vertex of the right angle of a right triangle, having a given hypotenuse.

Ex. 212. The mid-points of the chords drawn through a given point on the circle.

Ex. 213. The mid-points of the chords drawn through a given point within the circle.

Ex. 214. A point at a given distance from a given point.

Ex. 215. A point at a given distance from a given straight line.

Ex. 216. A point at a given distance from a given circle.

Ex. 217. A point equidistant from two given parallels.

Ex. 218. A point equidistant from two given intersecting lines.

Construct a circle of given radius,

Ex. 219. That will pass through two given points.

Ex. 220. That will pass through a given point and touch a given line.

Ex. 221. That will pass through a given point and touch a given circle.

Ex. 222. That will touch a given line and a given circle.

Construct a circle,

Ex. 223. That will pass through a given point and touch each of two given parallels.

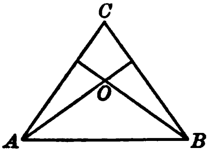
Ex. 224. That will touch a given circle and also touch a given line at a given point.

Ex. 225. That will touch a given line and also touch a given circle at a given point.

Ex. 226. That will pass through a given point and touch a given line at a given point.

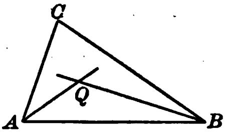
Ex. 227. That will pass through a given point and touch a given circle at a given point.

Ex. 228. That will touch two given intersecting lines and one of them at a given point.



Ex. 229. Find the locus of the orthocenter of a triangle, having a given base and a given vertical angle.

(*Show that* $\measuredangle O = \text{st.}\ \measuredangle - \measuredangle C$ *and apply* § 310.)



Ex. 230. Find the locus of the in-center of a triangle, having a given base and a given vertical angle.

(*Show that* $\measuredangle Q = \frac{1}{2} \measuredangle C + 90^\circ$ *and apply* § 310.)

Ex. 231. Draw the common tangents to two given circles.

BOOK III.

RATIO AND PROPORTION. SIMILAR FIGURES.

THE THEORY OF PROPORTION.

311. A **proportion** is an expression of the equality of two ratios.

It is written in one of the following forms:

$$\frac{a}{b}=\frac{c}{d};\ \ a:b::c:d;\ \ a:b=c:d.$$

This proportion is read "a is to b as c is to d"; or "the ratio of a to b is equal to the ratio of c to d."

312. The four quantities, a, b, c, and d, are called the **terms** of the proportion.

313. The first and the fourth terms are called the **extremes**; and the second and the third terms, the **means** of the proportion.

314. The first and the third terms of a proportion are called the **antecedents**; the second and the fourth terms, the **consequents**.

Thus, in the proportion, $a:b=c:d$, a and d are the **extremes**, b and c the **means**, a and c the **antecedents**, and b and d the **consequents**.

315. When the means of a proportion are equal, either mean is called the **mean proportional** between the first and the last terms, and the last term is called the **third proportional** to the first and the second terms.

Thus, in the proportion, $a:b=b:c$, b is the **mean proportional** between a and c, and c is the **third proportional** to a and b.

316. The fourth term of a proportion is called the **fourth proportional** to the other three, taken in order.

Thus, in the proportion, $a:b=c:d$, d is the fourth proportional to a, b, and c.

317. A series of equal ratios, as $a:b=c:d=e:f$, etc., is called a **continued proportion.**

318. The two terms of a ratio must be either quantities of the same kind, or the quantities must be represented by their numerical measures.

319. A proportion is arranged by **alternation** when antecedent is compared with antecedent and consequent with consequent.

320. A proportion is arranged by **inversion** when antecedents are made consequents and consequents made antecedents.

321. A proportion is arranged by **composition** when the sum of antecedent and its consequent is compared with either antecedent or consequent.

322. A proportion is arranged by **division** when the difference between antecedent and its consequent is compared with either antecedent or consequent.

323. A proportion is arranged by **composition and division** when the sum of antecedent and consequent is compared with the difference between antecedent and consequent.

PROPOSITION I. THEOREM.

324. *In any proportion the product of the extremes is equal to the product of the means.*

Given $a : b = c : d.$

To prove $ad = bc.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ § 311

Multiplying by bd,

$$ad = bc. \qquad \text{Q. E. D}$$

325. NOTE. In geometry the product of two quantities means the product of their numerical measures.

326. COR. 1. *If the antecedents of a proportion are equal, the consequents are equal, and conversely, if the consequents are equal, the antecedents are equal.*

327. COR. 2. *If the first three terms of a proportion are respectively equal to the first three terms of another proportion, their fourth terms are equal.*

Ex. 232. Find the value of x when $5 : 6 = x : 12$.

Ex. 233. Find the value of x when $x : 8 = 3 : 24$.

Ex. 234. Find the value of x when $11 : 44 = 15 : x$.

PROPOSITION II. THEOREM.

328. *If the product of two quantities is equal to the product of two others, either pair may be made the means and the other pair the extremes of a proportion.*

Given $ad = bc.$

To prove $a : b = c : d.$

Proof. Dividing both members of the given equation by bd,

$$\frac{a}{b} = \frac{c}{d},$$

or

$$a : b = c : d. \qquad \text{Q. E. D}$$

Ex. 235. Write two proportions formed from the equation $xy = ab$.

PROPOSITION III. THEOREM.

329. *The mean proportional between two quantities is equal to the square root of their product.*

Given $a : b = b : c$.

To prove $b = \sqrt{ac}$.

Proof. $b^2 = ac$. § 324

Extracting the square root of both members,

$b = \sqrt{ac}$. Q. E. D.

Ex. 236. Find the mean proportional between 16 and 49.

Ex. 237. Find the value of x when $4 : x = x : 9$.

Ex. 238. Find the value of x when $x : 6 = 24 : x$.

PROPOSITION IV. THEOREM.

330. *In any proportion the terms are in proportion by alternation.*

Given $a : b = c : d$.

To prove $a : c = b : d$.

Proof. $ad = bc$. § 324

$\therefore a : c = b : d$. § 328

Q. E. D.

PROPOSITION V. THEOREM.

331. *In any proportion the terms are in proportion by inversion.*

Given $a : b = c : d$.

To prove $b : a = d : c$.

Proof. $ad = bc$. § 324

$\therefore b : a = d : c$. § 328

Q. E. D.

PROPOSITION VI. THEOREM.

332. *In any proportion the terms are in proportion by composition.*

Given $a : b = c : d.$

To prove $a + b : a = c + d : c.$

Proof. $ad = bc.$ § 324

Adding ac to both members,

$$ac + ad = ac + bc.$$

Factoring, $a(c + d) = c(a + b).$

$\therefore a + b : a = c + d : c.$ § 328

Similarly, $a + b : b = c + d : d.$ Q. E. D.

PROPOSITION VII. THEOREM.

333. *In any proportion the terms are in proportion by division.*

Given $a : b = c : d.$

To prove $a - b : a = c - d : c.$

Proof. $ad = bc.$ § 324

Subtracting each member from ac,

$$ac - ad = ac - bc.$$

Factoring, $a(c - d) = c(a - b).$

$\therefore a - b : a = c - d : c.$ § 328

Similarly, $a - b : b = c - d : d.$ Q. E. D.

PROPOSITION VIII. THEOREM.

334. *In any proportion the terms are in proportion by composition and division.*

Given $a : b = c : d.$

To prove $a + b : a - b = c + d : c - d.$

Proof. $\frac{a+b}{a} = \frac{c+d}{c},$ (1) § 332

and $\frac{a-b}{a} = \frac{c-d}{c}.$ (2) § 333

Dividing (1) by (2), $\frac{a+b}{a-b} = \frac{c+d}{c-d},$

or $a + b : a - b = c + d : c - d.$ Q. E. D.

PROPOSITION IX. THEOREM.

335. *In a continued proportion the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Given $a : b = c : d = e : f.$

To prove $a + c + e : b + d + f = a : b.$

Proof. $ab = ab.$ Iden.

$ad = bc.$ § 324

$af = be.$ § 324

Adding these equations,

$$ab + ad + af = ab + bc + be.$$

Factoring, $a(b + d + f) = b(a + c + e).$

$\therefore a + c + e : b + d + f = a : b.$ § 328

Q. E. D.

PROPOSITION X. THEOREM.

336. *If the terms of a proportion are multiplied by the corresponding terms of another proportion, the products are in proportion.*

Given $a : b = c : d$, and $e : f = g : h$.

To prove $ae : bf = cg : dh$.

Proof. $\frac{a}{b} = \frac{c}{d}$. Hyp

$\frac{e}{f} = \frac{g}{h}$. Hyp.

Multiplying these equations,

$$\frac{ae}{bf} = \frac{cg}{dh},$$

or $ae : bf = cg : dh$. Q. E. D.

PROPOSITION XI. THEOREM.

337. *In any proportion equimultiples of the terms are in proportion.*

Given $a : b = c : d$.

To prove $ma : mb = mc : md$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 311

Multiplying both members by m,

$$\frac{ma}{mb} = \frac{mc}{md},$$

or $ma : mb = mc : md$. Q. E. D.

PROPOSITION XII. THEOREM.

338. *In any proportion like powers of the terms are in proportion.*

Given $a : b = c : d$.

To prove $a^n : b^n = c^n : d^n$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 311

Raising both members to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n},$$

or $$a^n : b^n = c^n : d^n.$$ Q. E. D.

339. Cor. *In any proportion like roots of the terms are in proportion.*

Ex. 239. Given $a : b = c : d$, and $m : n = a : c$, to show that $m : n = b : d$.

Ex. 240. Given $a : b = c : d$, and $r : s = b : a$, to show that $r : s = d : c$.

Ex. 241. Given $a : b = c : d$, $m : n = b : d$, and $p : q = c : a$, to show that $m : n = q : p$.

Ex. 242. Transform the proportion $p : q = r : s$ so that r becomes the fourth term.

Ex. 243. Given $x + y : x = 6 : 5$, to find the value of $\frac{x}{y}$.

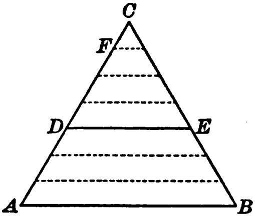
Ex. 244. Given $x - y : y = 5 : 4$, to find the value of $\frac{y}{x}$.

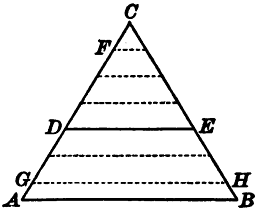
Ex. 245. Given $a : b = c : d$, to show that $ma : \frac{b}{n} = mc : \frac{d}{n}$.

Ex. 246. Given $a : x = x : a - x$, to find the value of x.

Proposition XIII. Theorem.

340. *A line parallel to one side of a triangle divides the other sides proportionally.*





Given the $\triangle$ ABC and the line DE $\parallel$ AB.

To prove $$DA : CD = EB : CE.$$

Proof. CASE I. When DA and CD are commensurable.

Let CF be a common measure of CD and DA, and let it be contained m times in DA and n times in CD.

Then $$\frac{DA}{CD}=\frac{m}{n}. \qquad (1)$$

Through the several points of division on DA and CD draw ‖s to AB. § 154

Then EB will be divided into m parts and CE into n parts, all of which parts are equal. § 182

$$\therefore \frac{EB}{CE}=\frac{m}{n}. \qquad (2)$$

Comparing (1) and (2),

$$\frac{DA}{CD}=\frac{EB}{CE}.$$

CASE II. When DA and CD are incommensurable.

Let CD be divided into any number of equal parts, and let one of these parts be applied successively to DA as many times as possible. A certain number of these parts will extend to some point on DA, as G, leaving a remainder GA, less than one of the equal parts.

Draw $GH \parallel AB$. § 154

Then since DG and CD are commensurable,

$$\frac{DG}{CD}=\frac{EH}{CE}. \qquad \text{Case I.}$$

By indefinitely increasing the number of parts into which CD is divided, the unit of measure will be indefinitely diminished, and the remainder GA will approach zero as a limit.

Therefore DG approaches DA and EH approaches EB as limits.

Hence $\frac{DG}{CD}$ approaches $\frac{DA}{CD}$ as a limit, § 278

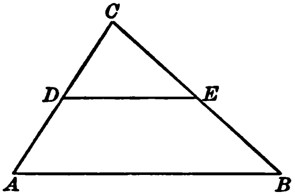
and $\frac{EH}{CE}$ approaches $\frac{EB}{CE}$ as a limit. § 278

But $\frac{DG}{CD}$ is constantly equal to $\frac{EH}{CE}$, Case I.

$$\therefore \frac{DA}{CD} = \frac{EB}{CE}, \qquad \S\ 275$$

(*if two variables are constantly equal, and each approaches a limit, their limits are equal*). Q. E. D.

341. Cor. 1. *If a line parallel to one side of a triangle cuts the other two sides, either side is to one of its segments as the other side is to its corresponding segment.*



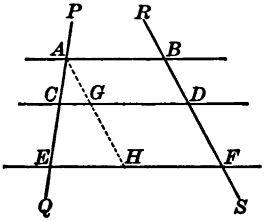
For $CD : DA = CE : EB$. § 340

$\therefore CD + DA : CD = CE + EB : CE,$ § 332

or $CA : CD = CB : CE.$

Similarly $CA : DA = CB : EB.$

342. Cor. 2. *If two straight lines are cut by parallels, the corresponding intercepts are proportional.*



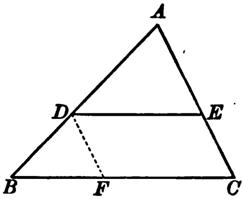
For let the straight lines PQ and RS be cut by the ||s AB, CD, and EF, and draw $AH \parallel RS$ (§ 154), cutting CD at G and EF at H.

Then $$\frac{AC}{CE}=\frac{AG}{GH}.$$ § 340

But $AG=BD$ and $GH=DF.$ § 169

$$\therefore \frac{AC}{CE}=\frac{BD}{DF}.$$

343. Cor. 3. *If a straight line parallel to the side BC of a triangle ABC cuts AB at D and AC at E, then* $DE:BC=AD:AB$.



For draw $DF \parallel AC$. § 154

Then $CF:BC=AD:AB.$ § 341

But $CF=DE.$ § 169

Substituting DE for its equal CF,

$$DE:BC=AD:AB.$$

Ex. 247. In a triangle ABC, DE is parallel to AB. Given $AC=10$, $BC=14$, $CD=6$, to find CE.

Ex. 248. In the triangle ABC, DE being parallel to AB, if $AD=8$, $DC=6$, and $BE=10$, find CE.

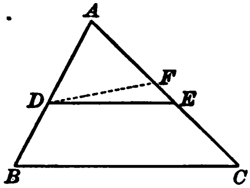
Ex. 249. In the triangle ABC, DE being parallel to AB, if $CD=\frac{2}{3}DA$ and $BC=30$, find CE.

Ex. 250. If in the triangle ABC, in which DE is parallel to AB, $CD=\frac{2}{3}CE$, $CD=\frac{5}{6}AD$, and $BE=18$, find CD.

Ex. 251. Draw through the point P in the arc AB a chord which shall be bisected by the chord AB.

PROPOSITION XIV. THEOREM.

344. *A straight line which divides two sides of a triangle proportionally is parallel to the third side* (converse of § 340).



Given the $\triangle$ ABC, and the line DE cutting AB and AC at D and E, respectively, making $\frac{AB}{AD}=\frac{AC}{AE}$.

To prove $DE \parallel BC$.

Proof. Suppose DF is drawn $\parallel BC$.

Then $\frac{AB}{AD}=\frac{AC}{AF}$. (1) § 341

But $\frac{AB}{AD}=\frac{AC}{AE}$. (2) Hyp.

Comparing (1) and (2),

$$\frac{AC}{AF}=\frac{AC}{AE}.$$

$\therefore AF=AE$. § 326

$\therefore DF$ and DE coincide, and hence $DE \parallel BC$. Q.E.D.

Ex. 252. In the figure for Proposition XIV, if $AD=4$, $BD=6$, $AE=6$, and $EC=8$, is DE parallel to BC?

Ex. 253. To a given circle draw two tangents that shall include an angle equal to a given angle.

HARMONIC DIVISION.

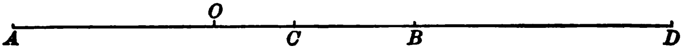
DEFINITIONS.

345. The line *AB* is divided **internally** at *C* when the point *C* is between the extremities of the line. *AC* and *BC* are **internal segments** of the line *AB*.

346. The line *AB* is divided **externally** at *D* when the point *D* is on the prolongation of *AB*. *AD* and *BD* are **external segments** of the line *AB*.

347. The line *AB* is equal to the **sum** of its internal segments and equal to the **difference** of its external segments.

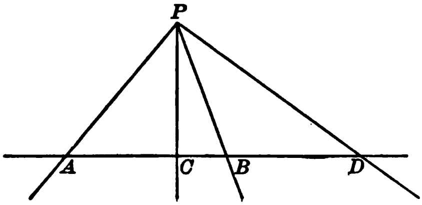
348. When a line is divided internally and externally in the same ratio, it is said to be divided **harmonically**.



Thus, if *AB* is divided internally at *C* and externally at *D* so that $\frac{AC}{BC}=\frac{AD}{BD}$, *AB* is said to be divided harmonically at the points *C* and *D*.

349. The four points *A*, *C*, *B*, and *D* are called **harmonic points**.

350. The four lines from any point, *P*, through the harmonic points form a **harmonic pencil**. *P* is the **vertex** of the pencil.



351. The points C and D are called **harmonic conjugates** with respect to A and B, and A and B harmonic conjugates with respect to C and D.

352. AB is the **harmonic mean** between AC and AD.

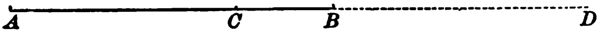
353. If AB is divided harmonically at C and D, then CD is divided harmonically at A and B. For if $\frac{AC}{BC}=\frac{AD}{BD}$, then by inversion and alternation $\frac{BC}{BD}=\frac{AC}{AD}$.

354. *Relation of harmonic points on a line.* Let A, C, B, D, be harmonic points. Since $\frac{AC}{BC}=\frac{AD}{BD}$, it follows that if AC is greater than BC, AD is greater than BD; that is, two conjugate points cannot both lie between the other conjugate points: and if C, the conjugate of D, lies between B and O, the mid-point of AB, D lies in the prolongation of AB, and, similarly, if C lies between A and O, D lies in the prolongation of BA.

Again, as C is made to approach one of the points, as B, the ratio $\frac{AC}{BC}$ increases, therefore the ratio $\frac{AD}{BD}$, its equal, increases. Hence, as the ratio $\frac{AC}{BC}$ becomes indefinitely large, or as the segment BC becomes indefinitely small, as compared with the segment AC, BD becomes indefinitely small as compared with AD; that is, as C approaches B, D approaches B.

Finally, as C approaches O, the mid-point of AB, the ratio $\frac{AC}{BC}$ approaches unity; therefore, since C is taken between A and B, D must be in the prolongation of AB or of BA. Since $AD=BD+AB$ or $BD-AB$, as $\frac{AD}{BD}$ approaches unity BD approaches infinity. Therefore, when C passes from a position between B and O to a position between A and O, D, the conjugate of C, changes from a position in the prolongation of AB to one in the prolongation of BA.

355. Theorem. *The harmonic mean between two magnitudes is equal to twice their product divided by their sum.*



Proof. Denoting these magnitudes by lines, let the line AB be divided harmonically at C and D.

Then $$\frac{AC}{BC}=\frac{AD}{BD}.\qquad \S\ 348$$

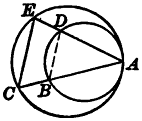
Whence $$AD\times BC=AC\times BD,$$

or $$AD\,(AB-AC)=AC\,(AD-AB).$$

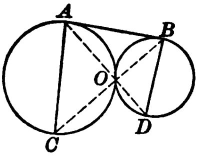
That is, $$AD\times AB-AD\times AC=AD\times AC-AC\times AB.$$

Whence $$AB\,(AC+AD)=2\,AC\times AD.$$

$$\therefore\ AB=\frac{2\,AC\times AD}{AC+AD}.$$ Q. E. D



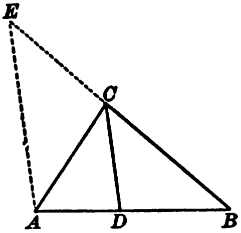
Ex. 254. If two circles are tangent internally, the chords of the greater, drawn from the point of contact, are divided proportionally by the smaller circle.



Ex. 255. If two polygons are congruent, their circumscribed circles are congruent.

PROPOSITION XV. THEOREM.

356. *The bisector of an interior angle of a triangle divides the opposite side internally into segments which are proportional to the adjacent sides.*



Given the $\triangle$ ABC and the bisector CD of the interior $\angle$ C.

To prove $AD : BD = AC : BC.$

Proof. Draw $AE \parallel CD$ to meet BC produced at E. § 154

Then in $\triangle ABE$, $AD : BD = CE : BC$. § 340

Also $\angle CAE = \angle ACD$, § 143

and $\angle CEA = \angle BCD$, § 148

But $\angle ACD = \angle BCD$. Hyp.

$\therefore \angle CAE = \angle CEA$. Ax. 1

$\therefore AC = CE$. § 103

Substituting AC for its equal, CE,

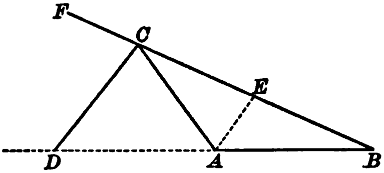
$AD : BD = AC : BC$. Q.E.D.

357. COR. *The line from the vertex of a triangle dividing the opposite side into segments proportional to the adjacent sides bisects that angle* (converse of § 356).

For only one point divides a given line internally in a given ratio; therefore, the point that divides a side of a triangle internally into segments proportional to the adjacent sides is in the bisector of the opposite angle.

PROPOSITION XVI. THEOREM.

358. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides* (correlative of § 356).



Given the $\triangle$ ABC, and the bisector CD of the exterior $\measuredangle$ C.

To prove $AD : BD = AC : BC.$

Proof. Draw $AE \parallel CD$ to meet BC at E. § 154

Then in $\triangle BDC$, $AD : BD = EC : BC.$ § 341

Also, $\measuredangle CAE = \measuredangle ACD,$ § 143

(alt. int. $\measuredangle$s of $\parallel$ lines),

and $\measuredangle CEA = \measuredangle DCF,$ § 148

(cor. $\measuredangle$s of $\parallel$ lines).

But $\measuredangle ACD = \measuredangle DCF.$ Hyp.

$\therefore \measuredangle CAE = \measuredangle CEA.$ Ax. 1

$\therefore AC = EC.$ § 103

Substituting AC for its equal, EC,

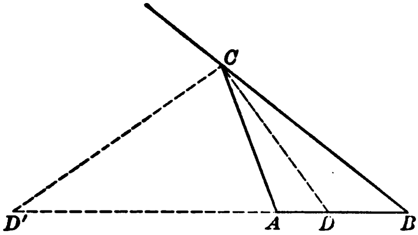
$AD : BD = AC : BC.$ Q. E. D.

Ex. 256. Divide one side of a triangle into segments proportional to the other two sides.

Ex. 257. In the figure for Proposition XV. if $AC = 8$, $BC = 20$, and $AD = 4$, find BD.

Ex. 258. In the same figure find AD, if $AB = 14$, $AC = 9$, and $BC = 12$.

359. Cor. *The bisectors of the interior and the exterior angle at one vertex of a triangle, not isosceles, divide the opposite side harmonically in the ratio of the adjacent sides.*



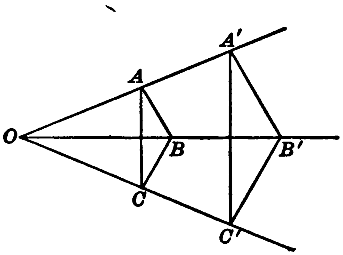
Given CD and CD′, the bisectors of the interior and the exterior ∡ C of the △ ABC.

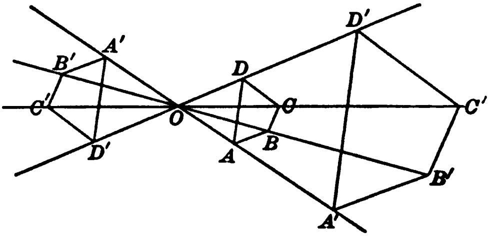
To prove $$\frac{AD}{BD}=\frac{AD'}{BD'}=\frac{AC}{BC}.$$

Proof. $$\frac{AD}{BD}=\frac{AC}{BC},\qquad (1)\quad \S\ 356$$

and $$\frac{AD'}{BD'}=\frac{AC}{BC}.\qquad (2)\quad \S\ 358$$

$$\therefore\ \frac{AD}{BD}=\frac{AD'}{BD'}=\frac{AC}{BC}.$$ Q. E. D.





SIMILAR FIGURES.

360. If any number of straight lines meet in a common point, as O, and on each of these lines two points, as A and A', B and B', C and C', are taken so that the ratios $\frac{OA}{OA'}$, $\frac{OB}{OB'}$, $\frac{OC}{OC'}$, etc., are all equal, the points A and A', B and B', C and C', etc., are called **homologous points**; the lines joining pairs of homologous points, as AB and $A'B'$, are called **homologous lines**; and the angles formed by pairs of homologous lines, as angles ABC and $A'B'C'$, are called **homologous angles**; and the figures whose sides are homologous lines and whose angles are homologous angles, as triangles ABC and $A'B'C'$, are called **similar figures.**

361. It is evident from § 151 that homologous angles are equal.

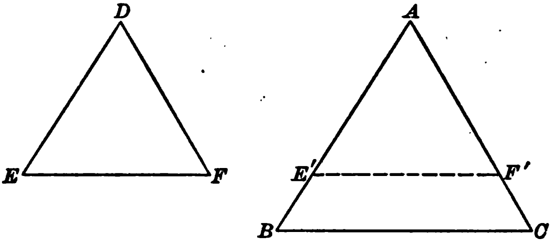
362. The point O is called the **center of similitude** of the similar figures, and the constant ratio $\frac{OA}{OA'}$, $\frac{OB}{OB'}$, etc., is called their **ratio of similitude.**

363. Since by § 343 $\frac{AB}{A'B'}=\frac{OA}{OA'}$, the ratio of similitude, it follows that the ratio of any two homologous sides of two similar polygons is equal to their ratio of similitude. In other words, the ratio of similitude of two similar polygons is the ratio of any two homologous lines of the polygons. Therefore, **similar polygons are those whose homologous angles are equal and whose homologous sides are proportional.**

364. When the ratio of similitude of two similar figures is unity, the center of similitude is called the **center of symmetry,** and the figures are said to be **symmetrical** with respect to that point as a center.

PROPOSITION XVII. THEOREM.

365. *Two triangles are similar if the angles of the one are equal, respectively, to the angles of the other.*



Given two $\triangle$s, ABC and DEF, having $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

To prove $\triangle ABC \sim \triangle DEF$.

Proof. Let $\triangle DEF$ be applied to $\triangle ABC$ so that $\angle D$ shall coincide with its equal, $\angle A$, and EF fall along $E'F'$.

Then $\angle AE'F' = \angle B$. Hyp.

$\therefore E'F' \parallel BC$. § 149

$\therefore AE' : AB = AF' : AC$, § 341

or $DE : AB = DF : AC$.

Again, $E'F' : BC = AE' : AB$, § 343

or $EF : BC = DE : AB$.

$\therefore \triangle ABC \sim \triangle DEF$, § 363

(having their $\angle$s equal and their homologous sides proportional).

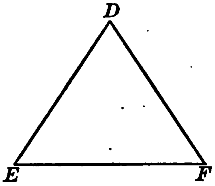
Q. E. D.

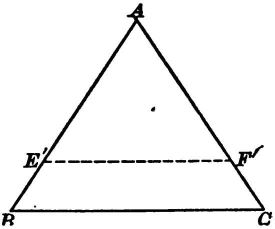
366. Cor. 1. *Two triangles are similar if two angles of the one are equal, respectively, to two angles of the other.*

367. Cor. 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

PROPOSITION XVIII. THEOREM.

368. *Two triangles are similar if they have an angle of the one equal to an angle of the other and the including sides proportional.*





Given $\triangle$s ABC and DEF in which $\measuredangle A = \measuredangle D$, and $\frac{AB}{DE} = \frac{AC}{DF}$.

To prove $\triangle ABC \sim \triangle DEF$.

Proof. Let $\triangle DEF$ be applied to $\triangle ABC$ so that $\measuredangle D$ shall coincide with its equal, $\measuredangle A$, and let EF take the position $E'F'$.

$$\frac{AE'}{AB} = \frac{AF'}{AC}. \qquad \text{Hyp.}$$

$$\therefore E'F' \parallel BC. \qquad \S\ 344$$

$$\therefore \measuredangle AE'F' = \measuredangle B, \qquad \S\ 148$$

or $$\measuredangle E = \measuredangle B.$$

$$\therefore \triangle ABC \sim \triangle DEF, \qquad \S\ 366$$

(*having two* $\measuredangle$*s of one = two* $\measuredangle$*s of the other*).

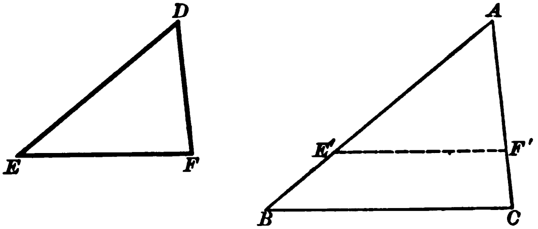
Q. E. D.

Ex. 259. The perpendicular from any point on a circle to a chord is the mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

Ex. 260. If AD and BE are altitudes of the triangle ABC, the triangles ACD and BCE are similar.

PROPOSITION XIX. THEOREM.

369. *Two triangles are similar if their sides are respectively proportional.*



Given △s ABC and DEF, in which $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

To prove $\triangle ABC \sim \triangle DEF$.

Proof. On AB lay off $AE' = DE$, and on AC lay off $AF' = DF$ and draw $E'F'$.

$\frac{AB}{AE'} = \frac{AC}{AF'}$. Hyp.

$\therefore E'F' \parallel BC$. § 344

$\therefore \angle AE'F' = \angle B$,

and $\angle AF'E' = \angle C$. § 148

$\therefore \triangle ABC \sim \triangle AE'F'$, § 366

Then $\frac{BC}{E'F'} = \frac{AB}{AE'} = \frac{AB}{DE}$. § 343

But $\frac{BC}{EF} = \frac{AB}{DE}$. Hyp.

$\therefore \frac{BC}{E'F'} = \frac{BC}{EF}$.

$\therefore E'F' = EF$. § 326

and $\triangle DEF \cong \triangle AE'F'$, § 123

But $\triangle ABC \sim \triangle AE'F'$. Just proved

$\therefore \triangle ABC \sim \triangle DEF$. Q.E.D

PROPOSITION XX. THEOREM.

370. *Two triangles are similar if the sides of the one are, respectively, parallel or perpendicular to the sides of the other.*

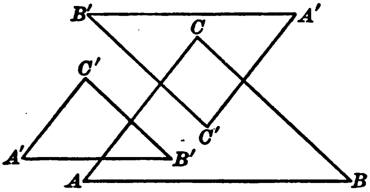


FIG. 1.

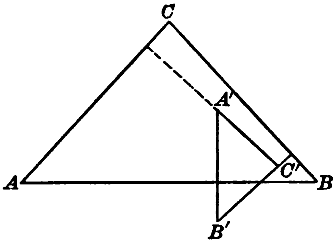


FIG. 2.

Given the △s ABC and A'B'C' in which the sides A'B', A'C', and B'C' are, respectively, parallel to AB, AC, and BC in Fig. 1, and perpendicular in Fig. 2.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. Since the sides of the two triangles in Fig. 1 are parallel, and in Fig. 2 perpendicular each to each, the angles formed by each pair of parallel or perpendicular sides are in both cases either equal or supplementary. §§ 150, 152

Hence, in both cases, there are three hypotheses:

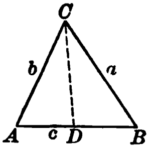
1. $\angle A + \angle A' = \text{st.} \angle$; $\angle B + \angle B' = \text{st.} \angle$; $\angle C + \angle C' = \text{st.} \angle$.
2. $\angle A = \angle A'$; $\angle B + \angle B' = \text{st.} \angle$; $\angle C + \angle C' = \text{st.} \angle$.
3. $\angle A = \angle A'$; $\angle B = \angle B'$; $\angle C = \angle C'$. § 159

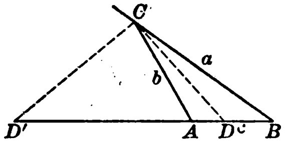
But neither the first nor the second hypothesis can be true, else the sum of the angles of the triangle would be greater than a straight angle, which is impossible. § 155

Therefore, the third hypothesis is true.

$\therefore \triangle ABC \sim \triangle A'B'C'$. § 365

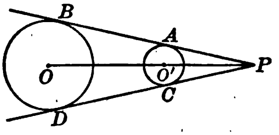
Q. E. D.

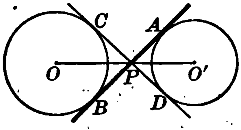




Ex. 261. In the triangle ABC, given $a = 15$, $b = 10$, and $c = 20$, to find the segments of c determined by the bisector of the interior angle at C.

Ex. 262. In the triangle ABC, given $a = 12$, $b = 9$, and $c = 4$, to find the internal and external segments of c determined by the interior and the exterior bisectors of the angle at C.



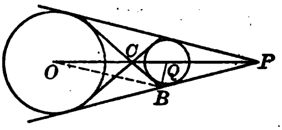


Ex. 263. The common exterior tangents to two circles intersect on the center-line of the circles.

Ex. 264. The common interior tangents to two circles intersect on the center-sect of the circles.

Ex. 265. The common exterior tangents of two circles are equal.

Ex. 266. The common interior tangents of two circles are equal.

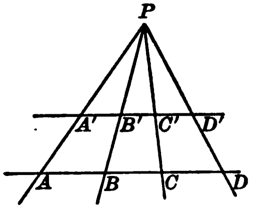


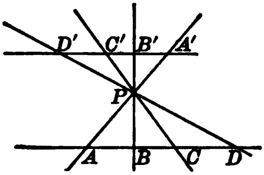
Ex. 267. The centers of two circles, the point of intersection of their common interior tangents, and the point of intersection of their common exterior tangents are harmonic points.

(*Draw BO and BQ and apply* § 258 *and* § 359.)

PROPOSITION XXI. THEOREM.

371. *If two parallels are cut by three or more concurrent transversals, the corresponding segments are proportional.*





Given the transversals concurrent at P and cutting the parallels AD and A'D' in A, B, C, D, and A', B', C', D'.

To prove $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}$.

Proof. Since $AD \parallel A'D'$,

$\triangle APB \sim \triangle A'PB'$, $\triangle BPC \sim \triangle B'PC'$, etc. § 370

In similar $\triangle$s APB and $A'PB'$,

$$\frac{AB}{A'B'}=\frac{PB}{PB'}. \quad (1) \quad § 363$$

Also in similar $\triangle$s PBC and $PB'C'$,

$$\frac{BC}{B'C'}=\frac{PB}{PB'}. \quad (2) \quad § 363$$

Comparing (1) and (2),

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

In like manner it is proved

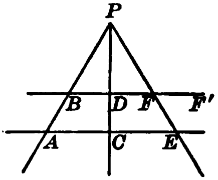
$$\frac{BC}{B'C'}=\frac{CD}{C'D'}.$$

$$\therefore \frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}.$$

Q. E. D.

PROPOSITION XXII. THEOREM.

372. *If three or more non-parallel straight lines intercept proportional segments on two parallels, they are concurrent.*



Given AB, CD, and EF, non-parallel lines cutting the ∥s AE and BF, making $\frac{AC}{BD} = \frac{CE}{DF}$.

To prove *AB, CD, and EF are concurrent.*

Proof. Produce AB and CD to meet at P, and suppose PE to cut BF at F'.

Then $$\frac{AC}{BD} = \frac{CE}{DF'}. \qquad \text{§ 371}$$

But $$\frac{AC}{BD} = \frac{CE}{DF}. \qquad \text{Hyp.}$$

$$\therefore \frac{CE}{DF'} = \frac{CE}{DF}.$$

Hence F' coincides with F, and EF' with EF, § 10 (*b*) or PE passes through F.

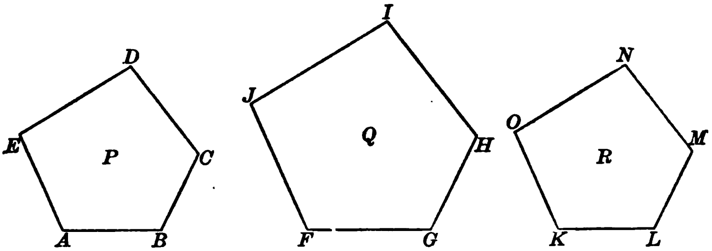
∴ AB, CD, and EF are concurrent. Q. E. D.

Ex. 268. In triangle ABC, EF is drawn parallel to AB. If $AC = 10$, $BC = 16$, $AB = 18$, and $EC = 3$, find EF and BF.

Ex. 269. The sides of a triangle are given, 12, 18, and 24 ft., respectively. In a similar triangle the shortest side is 8 ft. Find the other sides.

PROPOSITION XXIII. THEOREM.

373. *Two polygons similar to a third polygon are similar to each other.*



Given polygons Q and R, each similar to polygon P.

To prove $Q \sim R$.

Proof. ∡s K and F each = ∡ A. § 363

$\therefore$ ∡ K = ∡ F.

Similarly each ∡ of R = corresponding ∡ of Q.

Again, $\frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HI}$, etc. (1) § 363

Also $\frac{AB}{KL} = \frac{BC}{LM} = \frac{CD}{MN}$, etc. (2) § 363

Dividing (2) by (1),

$$\frac{FG}{KL} = \frac{GH}{LM} = \frac{HI}{MN}, \text{ etc.}$$

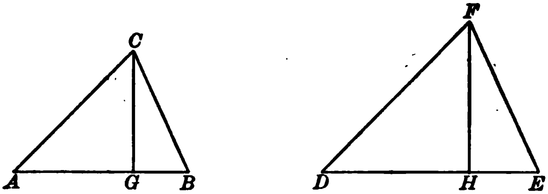
$\therefore Q \sim R$, § 363

(having homologous ∡s equal and homologous sides proportional).

Q. E. D.

PROPOSITION XXIV. THEOREM.

374. *The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.*



Given ABC and DEF two similar △s, and CG and FH two homologous altitudes.

To prove $$\frac{CG}{FH}=\frac{AC}{DF}.$$

Proof. In rt. △s ACG and DFH, $\measuredangle A=\measuredangle D$. § 363

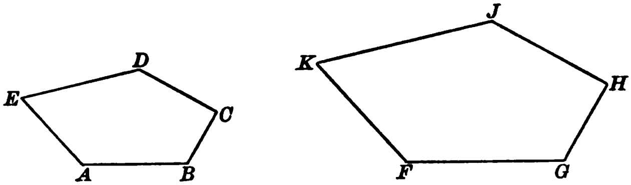
$\therefore$ rt △ $ACG \sim$ rt. △ DFH. § 367

$$\therefore \frac{CG}{FH}=\frac{AC}{DF}.$$ § 363

Q. E. D.

Proposition XXV. Theorem.

375. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



Given two similar polygons, ABCDE and FGHJK.

To prove $$\frac{AB+BC+CD+DE+EA}{FG+GH+HJ+JK+KF}=\frac{AB}{FG}.$$

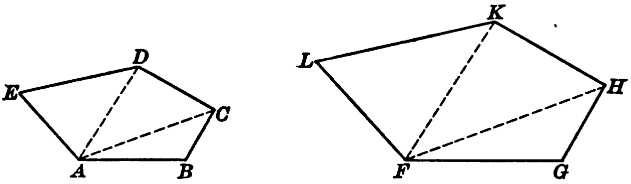
Proof. $$\frac{AB}{FG}=\frac{BC}{GH}=\frac{CD}{HJ}=\frac{DE}{JK}=\frac{EA}{KF}.$$ § 363

$$\therefore \frac{AB+BC+CD+DE+EA}{FG+GH+HJ+JK+KF}=\frac{AB}{FG}.$$ § 335

Q. E. D.

PROPOSITION XXVI. THEOREM.

376. *Two similar polygons may be divided into the same number of similar triangles, similarly placed.*



Given two similar polygons, ABCDE and FGHKL.

To prove $\triangle$*s* ABC, ACD, *etc., similar, respectively, to* $\triangle$*s* FGH, FHK, *etc.*

Proof. Since the polygons are similar,

$$\measuredangle B = \measuredangle G, \text{ and } \frac{AB}{FG} = \frac{BC}{GH}. \qquad \S\ 363$$

$$\therefore \triangle ABC \sim \triangle FGH. \qquad \S\ 368$$

Also $$\measuredangle BCD = \measuredangle GHK, \qquad (1)\ \S\ 363$$

and $$\measuredangle ACB = \measuredangle FHG. \qquad (2)\ \S\ 363$$

Subtracting (2) from (1),

$$\measuredangle ACD = \measuredangle FHK.$$

In similar $\triangle$s ABC and FGH,

$$\frac{AC}{FH} = \frac{AB}{FG}. \qquad \S\ 363$$

But $$\frac{AB}{FG} = \frac{CD}{HK}. \qquad \S\ 363$$

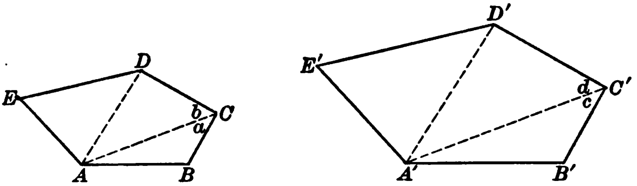
$$\therefore \frac{AC}{FH} = \frac{CD}{HK}.$$

$$\therefore \triangle ACD \sim \triangle FHK, \text{ etc.} \qquad \S\ 368$$

Q. E. D.

PROPOSITION XXVII. THEOREM.

377. *Two polygons are similar if they are composed of the same number of similar triangles, similarly placed* (converse of § 376).



Given polygons ABCDE and A'B'C'D'E', in which $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, etc.

To prove $ABCDE \sim A'B'C'D'E'$.

Proof. In similar $\triangle$s ABC and $A'B'C'$,

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}, \text{ and } \measuredangle a = \measuredangle c. \qquad \S\ 363$$

Also in similar $\triangle$s ACD and $A'C'D'$,

$$\frac{AC}{A'C'}=\frac{CD}{C'D'}, \text{ and } \measuredangle b = \measuredangle d. \qquad \S\ 363$$

$$\therefore \frac{AB}{A'B'}=\frac{CD}{C'D'}, \text{ and } \measuredangle (a+b) = \measuredangle (c+d), \text{ or } \measuredangle C = \measuredangle C'.$$

Similarly for the other sides and angles.

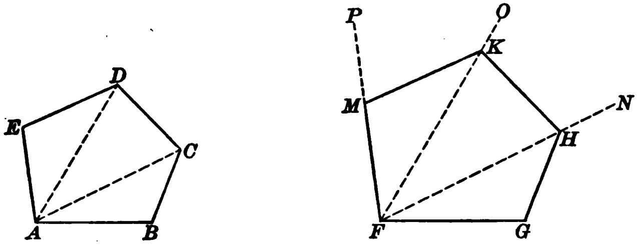
Therefore, the polygons have the angles equal and sides proportional, taken in the same order; hence they are similar. § 363

Q. E. D.

Ex. 270. In triangle ABC, given $AC = 11$ ft., and $BC = 12$ ft. On AC is laid off $AE = 8$ ft. and on BC is laid off $BF = 10$ ft., and EF is drawn. Is EF parallel to AB?

Proposition XXVIII. Problem.

378. *Upon a given line homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Given the line FG, homologous to AB of the given polygon ABCDE.

Required to construct on FG a polygon similar to ABCDE.

Construction. Draw the diagonals AC and AD. Also draw FN, FO, and FP, making $\angle$s GFN, NFO, and OFP equal, respectively, to $\angle$s BAC, CAD, and DAE. § 125

Draw GH, making $\angle G = \angle B$, and meeting FN at H.

Draw HK, making $\angle FHK = \angle ACD$, and meeting FO at K.

Draw KM, making $\angle FKM = \angle ADE$, and meeting FP at M. § 125

Then $FGHKM$ is the polygon required.

Proof. $\triangle FGH \sim \triangle ABC$; $\triangle FHK \sim \triangle ACD$, etc. § 366

$\therefore FGHKM \sim ABCDE$. § 377

Q. E. F.

379. Def. The **projection** of a straight line upon another straight line is the segment of the second line included between the perpendiculars drawn to it from the extremities of the first line.

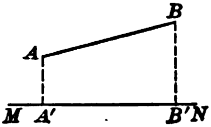


FIG. 1

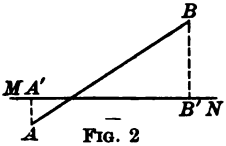


FIG. 2

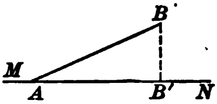
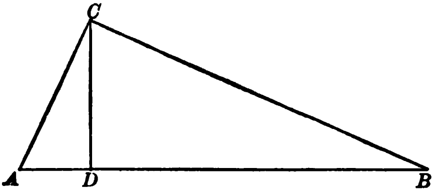


FIG. 3

In Figs. 1 and 2, $A'B'$ is the projection of AB on MN. In Fig. 3 where one extremity of AB is on MN, AB' is the projection of AB or MN.

PROPOSITION XXIX. THEOREM.

380. *In a right triangle the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse, and either leg is the mean proportional between the hypotenuse and its projection on the hypotenuse.*



Given the altitude CD upon the hypotenuse AB of the rt. △ ABC.

To prove I. $$\frac{AD}{CD}=\frac{CD}{BD}.$$

II. $$\frac{AB}{AC}=\frac{AC}{AD} \text{ and } \frac{AB}{BC}=\frac{BC}{BD}.$$

Proof. I. In rt. △ ADC, $\angle A$ and $\angle ACD$ are complements. § 160

But $\angle$s BCD and ACD are complements. Hyp.

$\therefore \angle A=\angle BCD.$ § 60

$$\therefore \triangle ACD \sim \triangle BCD. \qquad \S\ 367$$

$$\therefore \frac{AD}{CD}=\frac{CD}{BD}. \qquad \S\ 363$$

II. In rt. $\triangle$s ADC and ACB,

$$\measuredangle A = \measuredangle A. \qquad \text{Iden.}$$

$$\therefore \triangle ADC \sim \triangle ACB. \qquad \S\ 367$$

$$\therefore \frac{AB}{AC}=\frac{AC}{AD}. \qquad \S\ 363$$

Similarly, $$\frac{AB}{BC}=\frac{BC}{BD}. \qquad \text{Q. E. D.}$$

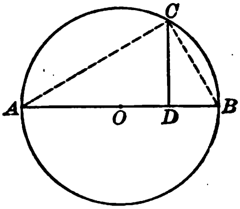
381. Cor. 1. *The squares of the legs of a right triangle have the same ratio as their projections on the hypotenuse.*

For $$\overline{AC}^2 = AB \times AD,$$

and $$\overline{BC}^2 = AB \times BD. \qquad \S\ 380$$

$$\therefore \frac{\overline{AC}^2}{\overline{BC}^2}=\frac{AD}{BD}.$$

382. Cor. 2. *The altitude upon the hypotenuse divides a right triangle into two similar triangles.*



383. Cor. 3. *If a perpendicular be drawn from any point in the circumference of a circle to a diameter,*

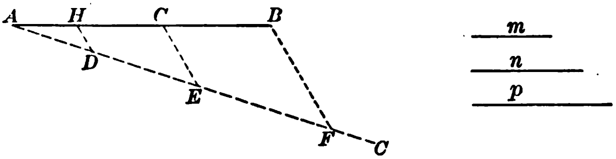
I. *The perpendicular is a mean proportional between the segments of the diameter.*

II. *The chord from that point to either extremity of the diameter is the mean proportional between the diameter and the projection of the chord on the diameter.*

384. Cor. 4. *In a right triangle the product of the hypotenuse and the altitude upon the hypotenuse is equal to the product of the two legs.*

Proposition XXX. Problem.

385. *To divide a given line into parts proportional to any number of given lines.*



Given the line AB and also the lines m, n, and p.

Required to divide AB into parts proportional to m, n, and p.

Construction. Draw AC making any convenient angle with AB, and on AC lay off AD, DE, and EF, equal, respectively, to m, n, and p.

Draw FB, and through D and E draw DH and $EG \parallel FB$. § 154

Then AB is divided as required.

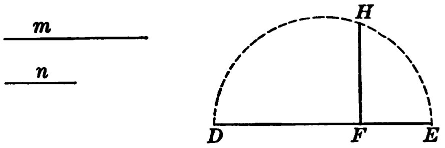
Proof. $$\frac{AH}{AD}=\frac{HG}{DE}=\frac{GB}{EF}.$$ § 342

That is, $$\frac{AH}{m}=\frac{HG}{n}=\frac{GB}{p},$$

or $$AH : HG : GB : m : n : p.$$ Q. E. F.

PROPOSITION XXXI. PROBLEM.

386. *To construct the mean proportional between two given lines.*



Given the lines m and n.

Required to construct the mean proportional between m and n.

Construction. On an indefinite line lay off $DF = m$ and $FE = n$. On DE as a diameter construct a semicircle. Draw $FH \perp DE$, meeting the semicircle at H. § 113

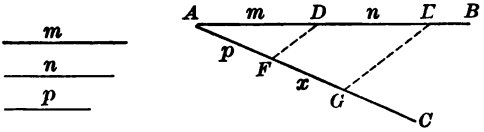
Then FH is the mean proportional required.

Proof. $DF : FH = FH : FE$. § 383

Q. E. F.

PROPOSITION XXXII. PROBLEM.

387. *To construct the fourth proportional to three given lines.*



Given the three lines m, n, and p.

Required to construct the fourth proportional to the three lines.

Construction. Draw the indefinite lines, AB and AC, making any convenient angle with each other.

On AB lay off $AD = m$ and $DE = n$. On AC lay off $AF = p$ and through E draw a parallel to DF, cutting AC at G. § 154

Then FG is the fourth proportional required.

Proof. $AD : DE = AF : FG$, § 342

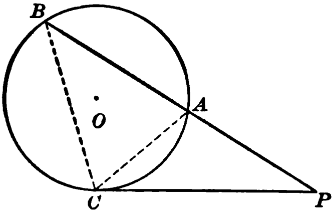
or $m : n :: p : FG$. Q. E. F

388. Cor. *In the preceding construction, if* $AF = DE = n$, *then* FG *is the third proportional to* m *and* n.

389. Def. A **secant** from a point to a circle is understood to be the segment of the secant between the point and the second point of intersection of the secant and the circle.

Proposition XXXIII. Theorem.

390. *If from a point without a circle a tangent and a secant are drawn, the tangent is the mean proportional between the whole secant and its external segment.*



Given the secant PAB and the tangent PC to the ⊙ O.

To prove $\dfrac{PB}{PC} = \dfrac{PC}{PA}$.

Proof. Draw AC and BC.

In △s BCP and ACP, $\angle P = \angle P$, Iden.

and $\angle ABC = \angle ACP$, §§ 288, 292

(each measured by ½ arc AC*).*

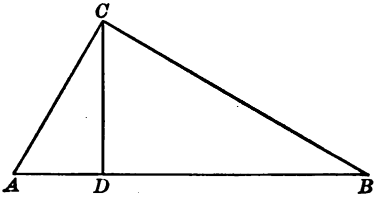
$\therefore \triangle BCP \sim \triangle ACP$. § 366

$\therefore \dfrac{PB}{PC} = \dfrac{PC}{PA}$. § 363

Q. E. D.

PROPOSITION XXXIV. THEOREM.

391. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two legs.*



Given the rt. △ ABC, ∡ C being the rt. ∡.

To prove $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Proof. Draw $CD \perp AB$. § 114

Then $AB \times AD = \overline{AC}^2$,

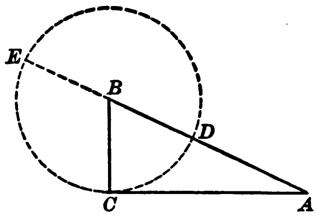
and $AB \times BD = \overline{BC}^2$. § 380

By addition, $AB(AD + BD) = \overline{AC}^2 + \overline{BC}^2$

or, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$ Q. E. D.

SECOND METHOD.

Given the rt. △ ABC.



With B as a center and the leg BC as a radius, describe a ⊙, cutting AB at D and AB produced at E.

This ⊙ is tangent to BC at the point C. § 252

Then $AE \times AD = \overline{AC}^2$, § 390

or $(AB + BC) \times (AB - BC) = \overline{AC}^2$.

Whence $\overline{AB}^2 - \overline{BC}^2 = \overline{AC}^2$,

or $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$. Q. E. D.

392. Cor. 1. *The square of either leg of a right triangle is equal to the difference of the square of the hypotenuse and the square of the other leg.*

393. Cor. 2. *The diagonal and the side of a square are incommensurable* (diag. : side $= \sqrt{2} : 1$).

Ex. 271. Given the legs of a right triangle, 60 ft. and 91 ft., to find the hypotenuse.

Ex. 272. Given the hypotenuse 125 ft. and leg 117 ft., to find the other leg of a right triangle.

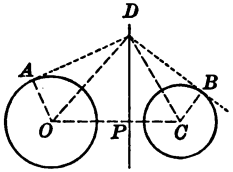
Ex. 273. If one leg of a right triangle is 25 ft. and the hypotenuse is 3 ft. longer than the other leg, what is the length of the hypotenuse?

Ex. 274. Given the perimeter of a right triangle, 56 ft. If the hypotenuse is 6 ft. less than the sum of the legs, what is the length of each side?

Ex. 275. In the right triangle ABC, AB being the hypotenuse, if $AC = 36$ and $BC = 77$, find the altitude CD.

Ex. 276. In the right triangle ABC, CD is drawn perpendicular to the hypotenuse. Given $CD = 12$ and $AB = 25$, to find AC.

Ex. 277. Construct the locus of the points from which tangents to two given circles are equal.



$$\overline{OP}^2 - \overline{CP}^2 = \overline{OA}^2 - \overline{CB}^2,$$

or

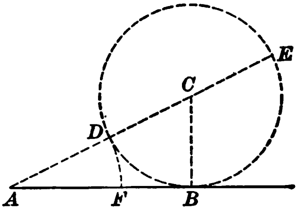
$$OC : R + r = R - r : OP - CP.$$

Ex. 278. Find the point from which tangents to three given circles are equal.

394. Def. A line is said to be divided in **extreme and mean ratio** when one of the segments is the mean proportional between the whole line and the other segment.

PROPOSITION XXXV. PROBLEM.

395. *To divide a given line in extreme and mean ratio.*



Given the line AB.

Required to divide AB in the extreme and mean ratio.

Construction. Draw $BC \perp AB$ and $= \frac{1}{2} AB$. § 290

With center C, and radius CB, describe a ⊙. This ⊙ is tangent to AB at B. § 252

Draw AC cutting the ⊙ at D and E.

On AB lay off $AF = AD$.

Then AB is divided at F as required.

Proof. $AE : AB = AB : AD$. § 390

Hence $AE - AB : AB = AB - AD : AD$. § 333

But $AB = DE$, and $AD = AF$. Cons.

$\therefore AE - DE : AB = AB - AF : AF$,

or $AF : AB = BF : AF$,

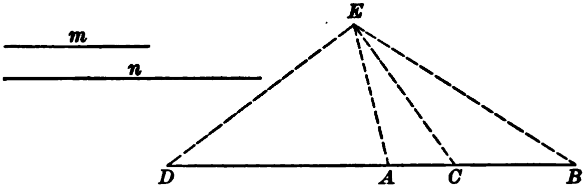
or $AB : AF = AF : BF$. § 331

Q.E.F.

396. NOTE. AB is said to be divided **internally** at F in extreme and mean ratio. If on BA produced AF' is laid off equal to AE, it is proved in like manner that $AB : AF' = AF' : BF'$; that is, AB is divided **externally** at F' in extreme and mean ratio.

PROPOSITION XXXVI. PROBLEM.

397. *To divide a given line harmonically in a given ratio.*



Given the line AB and the two lines, m and n.

Required to divide AB harmonically in the ratio $\frac{m}{n}$.

Construction. With A and B as centers and with radii having the given ratio $\frac{m}{n}$, describe arcs intersecting at E.

Draw AE and BE.

Bisect the interior and the exterior angle at E, § 129 the bisectors meeting BA at C and BA produced at D, respectively.

Then AB is divided at C and D as required.

Proof. $$\frac{AC}{BC}=\frac{AD}{BD}=\frac{AE}{BE}.$$ § 359

But $$\frac{AE}{BE}=\frac{m}{n}.$$ Cons.

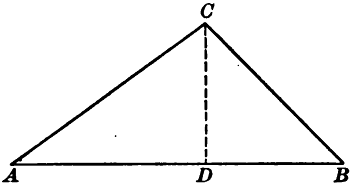
$$\therefore \frac{AC}{BC}=\frac{AD}{BD}=\frac{m}{n}.$$ Q.E.F.

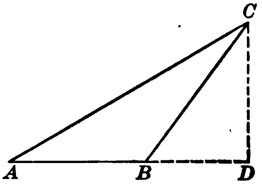
Ex. 279. Find the length of the greater segment of a line 10 ft. long, divided internally in extreme and mean ratio.

Ex. 280. If the length of the greater segment of a line divided internally in extreme and mean ratio is 6 ft., find the length of the line.

Proposition XXXVII. Theorem.

398. *In any triangle the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other upon that side.*





Given the △ ABC, $\measuredangle$ A being acute and CD ⊥ AB.

To prove $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AB \times AD$.

Proof. When $\measuredangle\, B$ is acute, $BD = AB - AD$.

When $\measuredangle\, B$ is obtuse, $BD = AD - AB$.

Squaring, in either case, $\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 - 2\,AB \times AD$.

Adding $\overline{CD}^2$ to each side,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{AD}^2 + \overline{CD}^2 - 2\,AB \times AD.$$

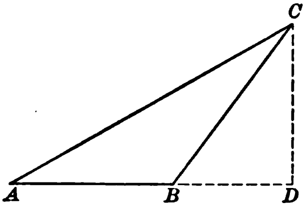
But $\overline{BD}^2 + \overline{CD}^2 = \overline{BC}^2$, and $\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2$. § 391

Substituting these values,

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AB \times AD.$$ Q.E.D.

Proposition XXXVIII. Theorem.

399. *In any triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of* *one of those sides by the projection of the other upon that side* (correlative of § 398).



Given the obtuse △ ABC, ∡ B being the obtuse ∡, and CD ⊥ AB produced.

To prove $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BD.$

Proof. $AD = AB + BD.$

$$\therefore \overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2 + 2\,AB \times BD.$$

Adding $\overline{CD}^2$ to both sides,

$$\overline{AD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{BD}^2 + \overline{CD}^2 + 2\,AB \times BD.$$

But $\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2$, and $\overline{BD}^2 + \overline{CD}^2 = \overline{BC}^2$. § 391

Substituting these values,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BD.$$ Q.E.D.

NOTE. By the principle of continuity Propositions XXXIV., XXXVII., and XXXVIII. may be stated as one theorem.

Ex. 281. Given the sides of a triangle, 12, 17, and 25 ft., to find the projection of each side upon each of the others.

Ex. 282. Given the sides of a triangle, 11, 13, and 20 ft., respectively, to find the projection of each side upon each of the others.

Ex. 283. Prove the converse of § 391.

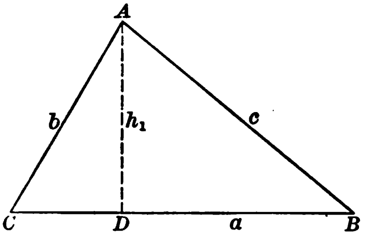
Ex. 284. If the sides of a triangle are 8, 15, and 17, respectively, is the triangle, right, acute, or obtuse ?

Ex. 285. If the sides of a triangle are respectively, 8, 12, and 14, is it right, acute, or obtuse ?

Ex. 286. If the sides of a triangle are 9, 12, and 18, is it right, acute, or obtuse ?

PROPOSITION XXXIX. PROBLEM.

400. *To compute the altitudes of a triangle in terms of its sides.*



Given the sides a, b, and c, of the △ ABC.

Required to find the altitudes, h_1, h_2, h_3, in terms of a, b, and c.

Solution. Let ∡ C be acute and let altitude $AD = h_1$.

$$c^2 = a^2 + b^2 - 2\,a \times CD. \qquad \text{§ 398}$$

Transposing, $2\,a \times CD = a^2 + b^2 - c^2$.

Whence $$CD = \frac{a^2 + b^2 - c^2}{2\,a}.$$

But $\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2 = (AC + CD)(AC - CD)$, § 392

or $$h_1^2 = \left(b + \frac{a^2 + b^2 - c^2}{2\,a}\right)\left(b - \frac{a^2 + b^2 - c^2}{2\,a}\right)$$

$$= \frac{(2\,ab + a^2 + b^2 - c^2)(2\,ab - a^2 - b^2 + c^2)}{4\,a^2}$$

$$= \frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{4\,a^2}$$

$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c+b-a)}{4\,a^2}.$$

Let $a + b + c = 2\,s$.

Then $a + b - c = 2\,(s - c)$,

$a + c - b = 2\,(s - b)$,

and $b + c - a = 2\,(s - a)$.

$$\therefore h_1^2 = \frac{2s \times 2(s-a) \times 2(s-b) \times 2(s-c)}{4a^2}$$

$$= \frac{16s(s-a)(s-b)(s-c)}{4a^2}.$$

$$\therefore h_1 = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly for h_2 and h_3. Q. E. F.

Ex. 287. Find the altitudes of the triangle whose sides are, respectively, 13, 14, and 15 ft.

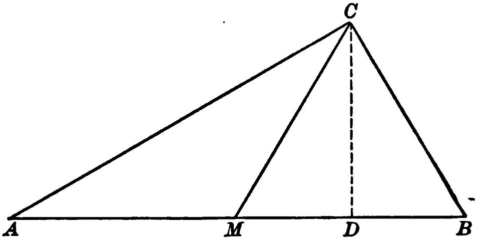
Ex. 288. Find the altitudes of a triangle whose sides are, respectively, 13, 20, and 21 ft.

Ex. 289. Any two altitudes of a triangle are inversely proportional to the sides to which they are drawn.

Ex. 290. In two similar triangles, ABC and $A'B'C'$, the altitude of ABC to the side AB is 19 ft. and the homologous altitude of $A'B'C'$ is 57 ft. If the perimeter of ABC is 185 ft., what is the perimeter of $A'B'C'$?

PROPOSITION XL. THEOREM.

401. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, increased by twice the square of the median to that side.*



Given the △ ABC and the median CM to the side AB.

To prove $\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AM}^2 + 2\,\overline{CM}^2.$

Proof. Draw the altitude CD. § 114

Then $\overline{AC}^2 = \overline{AM}^2 + \overline{CM}^2 + 2\,AM \times MD$. (1) § 399

and $\overline{BC}^2 = \overline{BM}^2 + \overline{CM}^2 - 2\,BM \times MD$. (2) § 398

But $BM = AM$. Hyp.

Adding (1) and (2) and substituting AM for BM,

$$\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AM}^2 + 2\,\overline{CM}^2.$$ Q.E.D

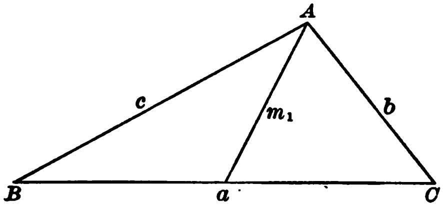
402. Cor. 1. *The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.*

Proof. Subtracting (2) from (1),

$$\overline{AC}^2 - \overline{BC}^2 = 4\,BM \times MD = 2\,AB \times MD.$$

Proposition XLI. Problem.

403. *To compute the medians of a triangle in terms of its sides.*



Given the sides of a △, a, b, and c, to find the medians, m₁, m₂, and m₃.

$$b^2 + c^2 = 2\,m_1^2 + 2\left(\frac{a}{2}\right)^2 = 2\,m_1^2 + \frac{a^2}{2}.$$ § 401

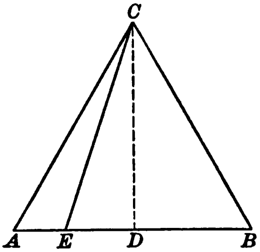
$$4\,m_1^2 = 2\,(b^2 + c^2) - a^2.$$

Whence $m_1 = \frac{1}{2}\sqrt{2\,(b^2 + c^2) - a^2}$.

Similarly for m_2 and m_3. Q.E.F.

PROPOSITION XLII. THEOREM.

404. *The square of a leg of an isosceles triangle is equal to the square of a line from the vertex of the vertical angle to the base, increased by the product of the segments into which it divides the base.*



Given the isosceles △ ABC, and the line CE drawn to any point in the base.

To prove $\overline{AC}^2 = \overline{CE}^2 + BE \times AE.$

Proof. Draw $CD \perp AB$. § 114

Then $\overline{AC}^2 = \overline{CD}^2 + \overline{AD}^2,$

and $\overline{CE}^2 = \overline{CD}^2 + \overline{DE}^2.$ § 391

$$\therefore \overline{AC}^2 - \overline{CE}^2 = \overline{AD}^2 - \overline{DE}^2$$

$$= (AD + DE)(AD - DE)$$

$$= BE \times AE.$$

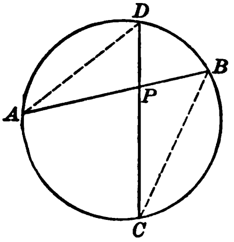
$$\therefore \overline{AC}^2 = \overline{CE}^2 + BE \times AE.$$ Q. E. D.

Ex. 291. Find the medians of a triangle whose sides are, respectively, 12, 14, and 16.

Ex. 292. Given the sides, a, b, c, of a triangle equal, respectively, 5, 29, and 30, to find the altitude and the median drawn to the side c.

Proposition XLIII. Theorem.

405. *If two chords intersect within a circle, the product of the segments of the one is equal to the product of the segments of the other.*



Given two chords, AB and CD, intersecting within the circle at P.

To prove $AP \times BP = CP \times DP$.

Proof Draw AD and BC.

In $\triangle$s APD and BPC,

$\measuredangle A = \measuredangle C$, and $\measuredangle D = \measuredangle B$. § 289

$\therefore \triangle APD \sim \triangle BPC$. § 366

$\therefore AP : CP = DP : BP$. § 363

Whence $AP \times BP = CP \times DP$. § 324

Q. E. D.

Ex. 293. The longest side of a triangle has the shortest corresponding altitude.

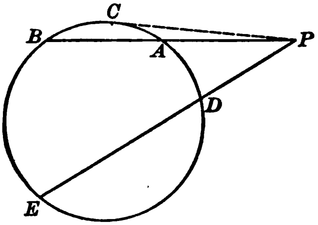
Ex. 294. The sum of the squares of the medians of a triangle is equal to three-fourths the sum of the squares of the sides.

Ex. 295. The sum of the squares of the sides of a triangle is equal to three times the sum of the squares of the lines from the centroid to the vertices.

Ex. 296. Given the leg of an isosceles triangle = 12 and the base = 15, to find the length of the line from the vertex to the base, making the segments of the base equal, respectively, to 4 and 11.

PROPOSITION XLIV. THEOREM.

406. *If from a point without a circle two secants are drawn, the product of one secant by its external segment is equal to the product of the other by its external segment.*



Given two secants, PAB and PDE.

To prove $PB \times PA = PE \times PD$.

Proof. Draw the tangent PC. § 309

Then $PB \times PA = \overline{PC}^2$,

and $PE \times PD = \overline{PC}^2$. § 390

$\therefore PB \times PA = PE \times PD$. Q. E. D.

NOTE. By the principle of continuity §§ 390, 405, and 406 may be stated as one theorem.

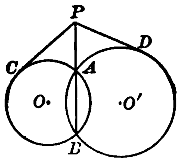
Ex. 297. Given the leg of an isosceles triangle, 14 ft., and the base 17 ft. What are the segments of the base made by a line 12 ft. long drawn from the vertex?

Ex. 298. Two chords, AB and CD, intersect within a circle at O. Given $AO = 8$, $BO = 4$, and $CD = 18$, to find DO and CO.

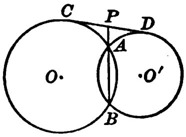
Ex. 299. Given two secants, PAB and PDE, to the same circle, in which $PA = 7$, $AB = 4$, $PD = 6$. Find DE.

Ex. 300. If a chord 5 ft. long is produced 4 ft., what is the length of the tangent drawn from its outer extremity?

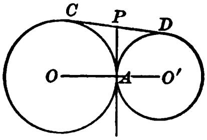
Ex. 301. The secant *PB* is four times its external segment, and the tangent from its outer extremity is 8 ft. Find the length of the secant.



Ex. 302. The tangents to two intersecting circles drawn from any point in their common chord produced are equal.



Ex. 303. The common chord of two intersecting circles, produced, bisects their common tangents.



Ex. 304. If two circles are tangent to each other, their common internal tangent bisects their common external tangents.

Ex. 305. Given the length of the tangent from a given point to a given circle = 6 ft., and the length of the secant from the same point = 9 ft. Find the length of the external segment of the secant.

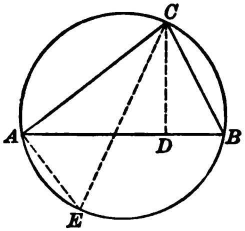
Ex. 306. Two circles have radii 30 in. and 78 in., respectively, and their center-sect is 73 in. Find the length of their common exterior tangent.

Ex. 307. Two circles have radii 16 and 20 ft., respectively, and their center-sect is 85 ft. Find the length of their common interior tangent.

Ex. 308. Given the sides of a triangle, 28, 65, and 89. Find the altitudes.

PROPOSITION XLV. THEOREM.

407. *The product of any two sides of a triangle is equal to the product of the altitude upon the third side by the diameter of a circumscribed circle.*



Given CE, the diameter of the circumscribed circle of the $\triangle$ ABC, and CD, the altitude upon AB.

To prove $AC \times BC = CE \times CD$.

Proof. Draw AE.

Then CAE is a rt $\angle$. § 290

In rt. $\triangle$s BCD and ACE,

$\angle B = \angle E$. § 289

$\therefore \triangle BCD \sim \triangle ACE$. § 367

$\therefore AC : CD = CE : BC$. § 363

Hence $AC \times BC = CE \times CD$. § 324

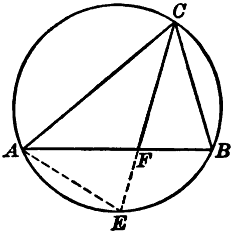
Q.E.D.

Ex. 309. The sum of the squares of the lines from any point on a circle to the vertices of an inscribed square is equal to twice the square of the diameter of the circle.

Ex. 310. The difference of the squares of two sides of a triangle is equal to the difference of the squares of their projections on the third side.

PROPOSITION XLVI. THEOREM.

408. *The square of the bisector of an angle of a triangle is equal to the product of the including sides diminished by the product of the segments of the third side determined by the bisector.*



Given the △ ABC, and CF, the bisector of the ∡ C.

To prove $\overline{CF}^2 = AC \times BC - AF \times BF.$

Proof. Circumscribe a circle about the △ *ABC* (§ 306), produce *CF* to meet the circle at *E*, and draw *AE*.

In △s *CFB* and *ACE*,

$$\angle BCF = \angle ACE. \qquad \text{Hyp.}$$

$$\angle B = \angle E. \qquad \text{§ 289}$$

$$\therefore \triangle CFB \sim \triangle ACE. \qquad \text{§ 366}$$

$$\therefore BC : CE = CF : AC. \qquad \text{§ 363}$$

Whence $AC \times BC = CF \times CE$ § 324

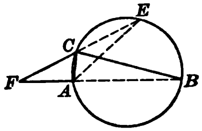
$$= CF(CF + FE)$$

$$= \overline{CF}^2 + CF \times FE.$$

But $CF \times FE = AF \times BF.$ § 405

$$\therefore AC \times BC = \overline{CF}^2 + AF \times BF,$$

or $\overline{CF}^2 = AC \times BC - AF \times BF.$ Q. E. D.



Ex. 311. The square of the bisector of an exterior angle of a triangle is equal to the product of the external segments determined by the bisector upon one of the sides, diminished by the product of the other two sides (correlative of § 408).

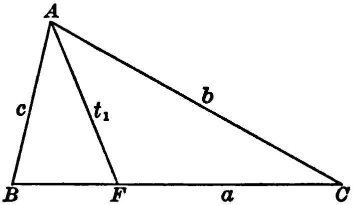
To prove $\overline{CF}^2 = AF \times BF - AC \times BC.$

Circumscribe a ⊙ about △ *ABC*, produce *CF* to meet the circle at *E*, and draw *AE*.

Prove △ *CFB* ∼ △ *ACE*, and proceed as in § 408.

PROPOSITION XLVII. PROBLEM.

409. *To compute the bisectors of a triangle in terms of its sides.*



Given the sides a, b, and c of a △, to find t_1, t_2, t_3, the bisectors, in terms of a, b, and c.

Solution. $t_1^2 = bc - BF \times FC.$ § 408

$BF : FC = c : b.$ § 356

$\therefore BF + FC : BF = b + c : c,$ § 332

or, $a : BF = b + c : c.$

Whence $BF = \dfrac{ac}{b+c}.$

Similarly $CF = \dfrac{ab}{b+c}.$

$$\therefore t_1^2 = bc - \frac{a^2bc}{(b+c)^2}$$

$$= bc\left[1 - \frac{a^2}{(b+c)^2}\right]$$

$$= \frac{bc[(b+c)^2 - a^2]}{(b+c)^2}$$

$$= \frac{bc[(b+c+a)(b+c-a)]}{(b+c)^2}$$

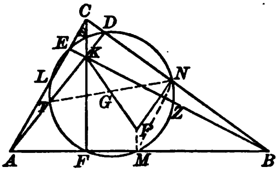
$$= \frac{bc[2\,s(2\,s-2\,a)]}{(b+c)^2}$$

$$= \frac{4\,bcs(s-a)}{(b+c)^2}.$$

$$\therefore t_1 = \frac{2}{b+c}\sqrt{bcs(s-a)}.$$

Similarly for t_2 and t_3. **Q.E.F.**

Ex. 312. **The circle through the mid-points of the sides of a triangle passes through the feet of the altitudes and bisects the lines from the orthocenter to the vertices. (The Nine-point Circle.)**



Let K be the orthocenter and P the circumcenter of $\triangle ABC$; L, M, N, the mid-points of the sides, and T, S, Z, the mid-points of AK, CK, and BK, respectively, and D, E, F, the feet of the altitudes.

Draw NM and NT.

$$\triangle NPM \sim \triangle AKC,$$

and $\triangle TGK \sim \triangle NGP.$ § 370

$NM = \frac{1}{2} AC.$ § 178

$\therefore NP = TK,\ GK = GP,$ and $GT = GN.$ § 363

$\therefore TG = \frac{1}{2} AP,$ radius circumscribed ⊙. § 178

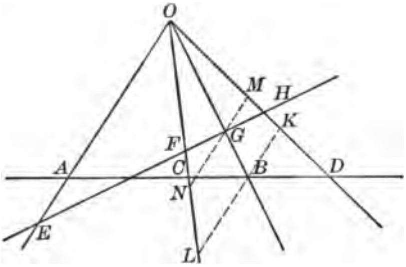
Hence with center G and radius $\frac{1}{2} AP$ a ⊙ will pass through T, the mid-point of AK, and, similarly, through S and Z.

Again, as $GN = GT$ the diameter of this ⊙ is TN and it passes through N, the mid-point of BC, and similarly through L and M.

Finally, this circle on TN, the hypotenuse of rt. $\triangle TDN$ as diameter, passes through D (§ 291), and similarly, through E and F.

PROPOSITION XLVIII. THEOREM.

410. *A harmonic pencil divides harmonically every line that it cuts.*



Given the harmonic pencil from O, cutting AD and EH, so that

$$\frac{AC}{BC} = \frac{AD}{BD}.$$

To prove $\frac{EF}{GF} = \frac{EH}{GH}.$

Proof. Through B and G draw KL and $MN \parallel OA$. § 154

Then $\triangle ACO \sim \triangle BCL$. § 370

$\therefore \frac{AC}{BC} = \frac{AO}{BL}$. § 363

$\triangle ADO \sim \triangle BDK$. § 370

$\therefore \frac{AO}{BK} = \frac{AD}{BD}$. § 363

But $\frac{AC}{BC} = \frac{AD}{BD}$. Hyp.

$\therefore \frac{AO}{BL} = \frac{AO}{BK}$. Ax. 1

$\therefore BL = BK$. § 326

Since $MN \parallel KL$ (§ 140), and $BL = BK$ (just proved),

$\therefore NG = MG$. § 371

Again $\triangle EFO \sim \triangle NFG.$ § 370

$\therefore \dfrac{EO}{NG} = \dfrac{EF}{GF}.$ (1) § 363

Also $\triangle EHO \sim \triangle GHM.$ § 370

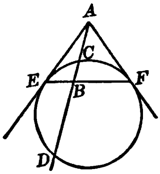
$\therefore \dfrac{EH}{GH} = \dfrac{EO}{MG},$ § 363

or $\dfrac{EH}{GH} = \dfrac{EO}{NG}.$. (2)

Comparing (1) and (2), $\dfrac{EF}{GF} = \dfrac{EH}{GH}.$ Q. E. D.

Ex. 313. If from a given point two tangents and a secant are drawn to a circle, the secant is divided harmonically by the point, the circle, and the chord of contact of the tangents.

Given the point A, the tangents AE and AF, and the secant ACD cutting EF at B.



To prove $\dfrac{AC}{BC} = \dfrac{AD}{BD}.$

Proof. $\overline{AE}^2 = AC \times AD.$ (1) § 390

$\overline{AE}^2 = \overline{AB}^2 + EB \times BF.$ § 404

But $EB \times BF = BC \times BD.$ § 405

$$\begin{aligned}
\therefore \overline{AE}^2 &= \overline{AB}^2 + BC \times BD \\
&= (AC + BC)^2 + BC \times BD \\
&= \overline{AC}^2 + 2\,AC \times BC + \overline{BC}^2 + BC \times BD \\
&= \overline{AC}^2 + AC \times BC + (AC + BC)BC + BC \times BD \\
&= \overline{AC}^2 + AC \times BC + AB \times BC + BC \times BD \\
&= \overline{AC}^2 + AC \times BC + AD \times BC. \qquad (2)
\end{aligned}$$

But $$AD = AC + BC + BD.$$

$$\therefore AC \times AD = AC(AC + BC + BD)$$
$$= \overline{AC}^2 + AC \times BC + AC \times BD. \quad (3)$$

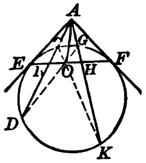
Substituting in (1) the values found in (2) and (3),

$$\overline{AC}^2 + AC \times BC + AD \times BC = \overline{AC}^2 + AC \times BC + AC \times BD.$$

Whence $$AD \times BC = AC \times BD,$$

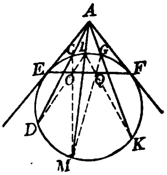
or $$\frac{AC}{BC} = \frac{AD}{BD}.$$

Ex. 314. If from a point A, without a circle, two secants, ACD and AGK, are drawn, the chords CK and DG intersect on the chord of contact of the tangents from the point A to the circle.



Proof. Let the chord of contact EF intersect CD at B, CK at O, and GK at H. Draw OA, OD, and OG.

By Ex. 313, A, C, B, D are harmonic points, and also A, G, H, K are harmonic points. Therefore, $O-ACBD$ and $O-AGHK$ are harmonic pencils. Therefore, since CO and OK form a straight line, DO and OG form a straight line. Therefore, the chords CK and DG intersect on EF.

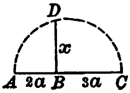


Ex. 315. If from a given point without a given circle any number of secants are drawn, the chords joining the points of intersection of the secants with the circle all cross on the same straight line.

Ex. 316. To draw a tangent from a given external point to a given circle by means of a ruler only.

(Draw any three secants to locate the chord of contact.)

Ex. 317. Given the line a, to construct $a\sqrt{6}$. (Construct mean proportional between $2a$ and $3a$.)



Ex. 318. Construct the expression $x = \sqrt{ab}$.

Ex. 319. Construct $x = \frac{ab}{c}$.

Ex. 320. Construct $x = \frac{abc}{de}$.

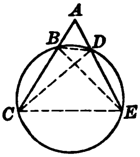
Ex. 321. Having given line a, construct $a\sqrt{\frac{3}{7}}$.

Ex. 322. Find the altitude of an equilateral triangle whose side is 10 ft.

Ex. 323. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of two opposite sides is equal to the sum of the squares of the other two sides.

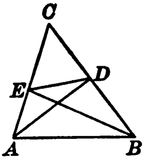
Ex. 324. Find the leg of an isosceles right triangle whose hypotenuse is 16 ft.

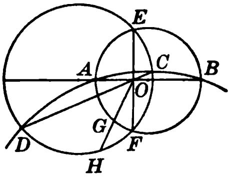
Ex. 325. If from a point A, without a circle, secants are drawn meeting the circle in B and C, and in D and E, respectively, the triangle ABE is similar to the triangle ACD, and the triangle ABD is similar to the triangle ACE.



Ex. 326. Given the sides of a triangle, a, b, and c, to find the sides of a similar triangle, if the side homologous to a is equal to d.

Ex. 327. If AD and BE are altitudes of the triangle ABC, the triangles DEC and ABC are similar.

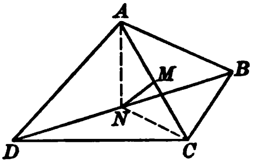




Ex. 328. If three circles intersect one another, the common chords are concurrent.

Let chords AB and CD meet at O and suppose EO produced meets the same two circles at G and H, respectively. Then apply § 405 to show that $OG = OH$, and, therefore, G and H coincide.

Ex. 329. The sum of the squares of the four sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the mid-points of the diagonals.

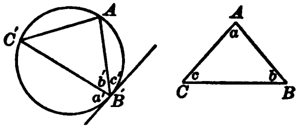


Ex. 330. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

(Special case of Ex. 329.)

Ex. 331. The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the lines that join the mid-points of the opposite sides.

Ex. 332. Find the greater segment of a line 12 ft. long, divided internally in extreme and mean ratio.

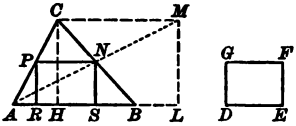


Ex. 333. Inscribe in a given circle a triangle similar to a given triangle.

Ex. 334. Circumscribe about a given circle a triangle similar to a given triangle.

Ex. 335. Draw from the vertex of the obtuse angle of a given obtuse triangle, to the opposite side, a line that shall be a mean proportional between the segments of that side.

Ex. 336. Find a point P in the arc AB such that the chord PA shall have a given ratio to the chord PB.



Ex. 337. To inscribe in a given triangle a rectangle similar to a given rectangle.

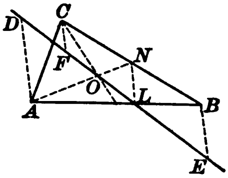
Let ABC be the given $\triangle$ and $DEFG$ the given rectangle. Construct $CHLM \sim DEFG$.

Ex. 338. Inscribe a square in a given triangle.

Ex. 339. Inscribe in a given semicircle a rectangle similar to a given rectangle.

Ex. 340. Inscribe a square in a given semicircle.

Ex. 341. If a line be drawn through the centroid of a triangle, and parallels be drawn from the vertices to this line, the one from one vertex is equal to the sum of the other two.



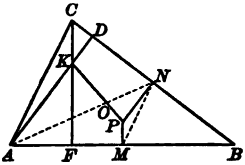
$$\triangle DAO \sim \triangle NOL. \quad AO = 2\,ON. \quad NL = \tfrac{1}{2}(CF + BE).$$

Ex. 342. The sum of the squares of the lines from any point within a rectangle to two opposite vertices is equal to the sum of the squares of the lines to the other opposite vertices.

Ex. 343. The perpendicular from the circumcenter to a side of a triangle is half the line from the opposite vertex to the orthocenter.

Ex. 344. The sum of the squares of the diagonals of a trapezoid is equal to the sum of the squares of the legs plus twice the product of the bases.

Ex. 345. The orthocenter, the centroid, and the circumcenter of a triangle are collinear, and the distance between the former two points is twice the distance between the latter two.



Let K be the orthocenter and P the circumcenter.

$\triangle MPN \sim \triangle AKC$. § 370

$MN = \frac{1}{2} AC$. § 178

$\therefore PN = \frac{1}{2} AK$. § 363

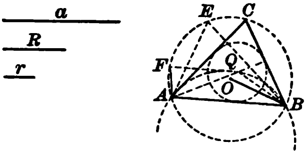
$\triangle OPN \sim \triangle AOK$ (370), and $PN = \frac{1}{2} AK$.

$\therefore NO = \frac{1}{2} AO$, and $PO = \frac{1}{2} OK$; hence $AO = \frac{2}{3} AN$, and therefore O is the centroid of ABC. § 205

Ex. 346. Construct a triangle, having given one side, the radius of the circumscribed circle, and the radius of the inscribed circle.

With any point O as center, and radius R, describe a circle. With any point A, on the circle as center and with radius $= a$, describe arc cutting the circle at B.

Draw any two chords, AE and BE, and on AB construct segment to contain $\frac{1}{2} \angle E +$ rt. $\angle$ (310). Draw $AF \perp AB$ and $= r$ (114). Through F draw $\parallel AB$ (154), cutting the arc at Q. With center Q, and radius r, describe a circle. Draw AC tangent to $\odot Q$. Draw BC. ABC is the $\triangle$ required.



For OA is radius of circumscribed $\odot$ by construction; AB is the side by construction, and $\odot Q$ is the inscribed circle by construction, and since $\angle C = \angle E$ (§ 289), and, by Ex. 230, the angle formed by the bisectors of $\angle$s A and $B = \frac{1}{2} (\angle C +$ rt. $\angle)$, BC is tangent to $\odot Q$; therefore $\odot Q$ is inscribed in $\triangle ABC$.

BOOK IV.

AREAS OF POLYGONS.

DEFINITIONS.

411. The **unit of surface** is a square whose side is the unit of length.

412. The **area of a surface** is the number of units of surface it contains.

413. Two plane figures are **equivalent** if their areas are equal.

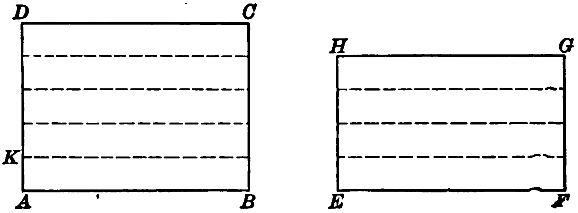
NOTE. The words "triangle," "rectangle," etc., are often used for "area of triangle," "area of rectangle," etc.

414. The side upon which a parallelogram is supposed to stand and the opposite side are called its **bases**.

415. The **altitude** of a parallelogram is the perpendicular distance between its bases.

PROPOSITION I. THEOREM.

416. *Rectangles having equal bases are to each other as their altitudes.*



Given two rectangles ABCD and EFGH, with equal bases, AB and EF.

To prove $ABCD : EFGH = AD : EH$.

Case I. When AD and EH are commensurable.

Proof. Let AK be a common measure of AD and EH which is contained m times in AD and n times in EH.

Then $$\frac{AD}{EH} = \frac{m}{n}. \tag{1}$$

Draw ⊥s to AD and EH at the several points of division. § 113

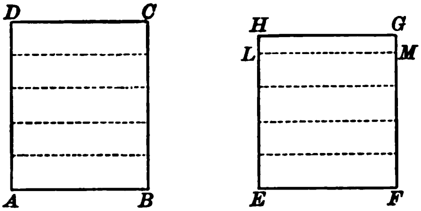
Then the rectangle $ABCD$ is divided into m rectangles and $EFGH$ into n rectangles, all of which are equal. § 177

$$\therefore \frac{ABCD}{EFGH} = \frac{m}{n}. \tag{2}$$

Comparing (1) and (2),

$$\frac{ABCD}{EFGH} = \frac{AD}{EH}.$$

Case II. When AD and EH are incommensurable.



Divide AD into any number of equal parts and apply one of these parts as a unit of measure to EH.

Since AD and EH are incommensurable, a certain number of these parts will extend from E to L, leaving a remainder HL less than one of the equal parts.

Draw $LM \perp EH$ (§ 113), meeting FG at M; then, because AD and EL are commensurable, $\frac{EFML}{ABCD} = \frac{EL}{AD}$. Case I.

But if the number of parts into which AD is divided be indefinitely increased, the remainder LH will approach zero as a limit.

Then the variable $\frac{EFML}{ABCD}$ will approach the limit $\frac{EFGH}{ABCD}$, and the variable $\frac{EL}{AD}$ will approach the limit $\frac{EH}{AD}$. § 278

But these variables are constantly equal, therefore their limits are equal. § 275

$$\therefore \frac{EFGH}{ABCD} = \frac{EH}{AD}.$$ Q. E. D.

417. Cor. *Because any side of a rectangle may be taken as the base, two rectangles having equal altitudes are to each other as their bases.*

Ex. 347. Find the bisectors of the triangle whose sides are, respectively, 18, 20, and 22 ft.

Ex. 348. Given the sides of a triangle ABC, $a = 16$, $b = 18$, $c = 20$. Find the bisector drawn to side a, the median drawn to side b, and the altitude drawn to side c.

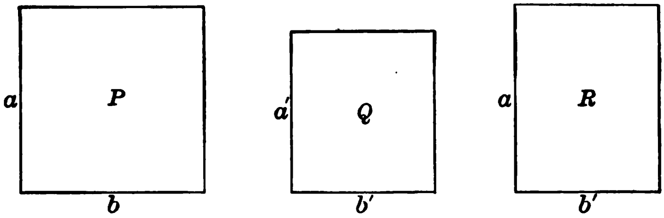
Ex. 349. The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.

Ex. 350. The radius of the nine-point circle of a triangle is half the radius of the circumscribed circle.

Ex. 351. The center of the nine-point circle of a triangle is collinear with the orthocenter and the circumcenter and midway between them.

PROPOSITION II. THEOREM.

418. *Two rectangles are to each other as the products of their bases and altitudes.*



Given two rectangles whose areas are P and Q with altitudes a and a' and bases b and b', respectively.

To prove $\frac{P}{Q}=\frac{ab}{a'b'}$.

Proof. Let R be a rectangle with altitude a and base b'.

Then $\frac{P}{R}=\frac{b}{b'}$, (1) § 416

and $\frac{R}{Q}=\frac{a}{a'}$. (2) § 417

Multiplying (1) by (2),

$$\frac{P}{Q}=\frac{ab}{a'b'}.$$

Q.E.D.

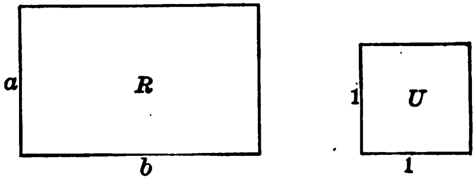
Ex. 352. The legs of a right triangle are 36 ft. and 77 ft. Find the legs of a similar right triangle whose hypotenuse is 170 ft.

Ex. 353. Describe a circle that shall pass through two given points and touch a given straight line.

Ex. 354. Two exterior tangent circles are tangent internally to a third circle. Prove that the perimeter of the triangle whose sides are their center-sects is equal to the diameter of the third circle.

PROPOSITION III. THEOREM.

419. *The area of a rectangle is equal to the product of its base and altitude.*



Given the rectangle R with base b and altitude a.

To prove $R = ab.$

Proof. Let the square U, each side of which is a unit of length, be taken as the unit of surface.

Then $\frac{R}{U} = \frac{a \times b}{1 \times 1}.$ § 418

Whence $R = ab \times U,$

or $R = ab$ units of surface;

i.e. $R = ab.$ Q. E. D.

Ex. 355. Find the area of a rectangle whose diagonal is 29 ft. and perimeter 82 ft.

Ex. 356. The area of a rectangle is 168 sq. ft. and its diagonal 25 ft. Find the perimeter.

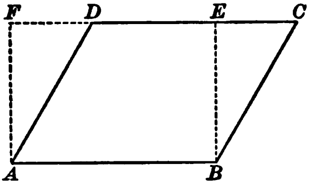
Ex. 357. Given the ratio of two consecutive sides of a rectangle, 5 : 11, and its area 495 sq. yd., to find the perimeter.

Ex. 358. The base of a rectangle is one foot greater than its altitude, and its diagonal is 29 ft. Find its area.

Ex. 359. A rectangular mirror 3 ft. by 2 ft. is surrounded by a frame 4 in. wide. What is the ratio of the area of the mirror to that of the frame?

Proposition IV. Theorem.

420. *The area of a parallelogram is equal to the product of its base and altitude.*



Given the ▱ ABCD whose altitude is BE.

To prove ▱ $ABCD = AB \times BE$.

Proof. Draw $AF \parallel BE$ to meet CD produced at F. § 154

In rt. △s AFD and BEC,

$AD = BC$ and $AF = BE$. § 167

$\therefore \triangle AFD \cong \triangle BEC$. § 122

From the trapezoid $ABCF$ take △ BEC, and the rectangle $ABEF$ remains.

From $ABCF$ take △ AFD, and the ▱ $ABCD$ remains.

∴ ▱ $ABCD$ = rectangle $ABEF = AB \times BE$. § 419

Q. E. D.

421. Cor. 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

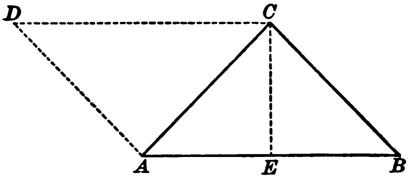
422. Cor. 2. *Parallelograms having equal bases are to each other as their altitudes.*

423. Cor. 3. *Parallelograms having equal altitudes are to each other as their bases.*

424. Cor. 4. *Parallelograms are to each other as the products of their bases and altitudes.*

PROPOSITION V. THEOREM.

425. *The area of a triangle is equal to half the product of its base and altitude.*



Given the $\triangle$ ABC and its altitude CE.

To prove $\triangle ABC = \frac{1}{2} AB \times CE.$

Proof. Construct the $\square$ $ABCD$. § 305

Area $\square ABCD = AB \times CE.$ § 420

But $\triangle ABC = \frac{1}{2} \square ABCD.$ § 166

$\therefore \triangle ABC = \frac{1}{2} AB \times CE.$ Q. E. D.

426. COR. 1. *A triangle is half a parallelogram of equal base and altitude.*

427. COR. 2. *Two triangles of equal bases are to each other as their altitudes.*

428. COR. 3. *Triangles of equal altitudes are to each other as their bases.*

429. COR. 4. *Two triangles are to each other as the products of their bases and altitudes.*

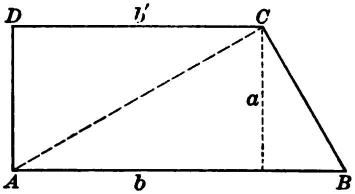
430. COR. 5. *Triangles of equal bases and altitudes are equivalent.*

Ex. 360. What is the side of the square whose area is equal to that of a rectangle 9 ft. by 16 ft.?

Ex. 361. A rectangle is twice as long as wide and its area is 512 sq. ft. Find the perimeter.

PROPOSITION VI. THEOREM.

431. *The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.*



Given the trapezoid ABCD, its bases b and b′, and its altitude a.

To prove area $ABCD = \frac{1}{2}a(b + b')$.

Proof. Draw the diagonal AC.

$\triangle ABC = \frac{1}{2}ab$,

and $\triangle ACD = \frac{1}{2}ab'$. § 425

$\therefore$ area $ABCD = \frac{1}{2}a(b + b')$. Q. E. D.

432. COR. *The area of a trapezoid is equal to the product of its median and altitude.*

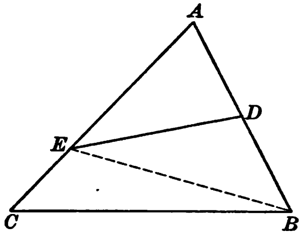
433. NOTE. The area of any polygon may be obtained by finding the sum of the areas of the triangles into which it may be divided by drawing diagonals from any one of its vertices, or by finding the sum of the areas of the triangles and trapezoids formed by drawing one diagonal and perpendiculars to it from the remaining vertices.

Ex. 362. The perimeter of a rectangle is 20 ft. greater than the sum of its diagonals, and its area is 660 sq. ft. Find the perimeter.

Ex. 363. If two homologous sides of two similar polygons are 11 and 12 ft., respectively, and the perimeter of the first is 66 ft., what is the perimeter of the second?

PROPOSITION VII. THEOREM.

434. *The areas of two triangles, having an angle in the one equal to an angle in the other, are to each other as the products of the sides including the equal angles.*



Given the $\triangle$s ABC and ADE with $\measuredangle$ A common.

To prove $$\frac{\triangle ABC}{\triangle ADE}=\frac{AB\times AC}{AD\times AE}.$$

Proof. Draw BE.

$\triangle$s ABE and ADE, having the common vertex E, and their bases, AB and AD in the same straight line, have the same altitude.

$$\therefore \frac{\triangle ABE}{\triangle ADE}=\frac{AB}{AD}. \qquad (1) \quad §\ 428$$

$\triangle$s ABC and ABE, having the common vertex B and their bases, AC and AE, in the same straight line, have the same altitude.

$$\therefore \frac{\triangle ABC}{\triangle ABE}=\frac{AC}{AE}. \qquad (2) \quad §\ 428$$

Multiplying (1) by (2),

$$\frac{\triangle ABC}{\triangle ADE}=\frac{AB\times AC}{AD\times AE}.$$ Q. E. D.

Ex. 364. What is the area of a parallelogram whose base is 35 ft. and altitude 24 ft.?

Ex. 365. The lines drawn from the mid-point of either diagonal of a quadrilateral to the extremities of the other divide the quadrilateral into two equivalent parts.

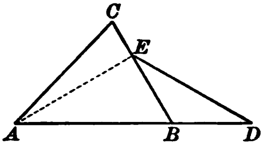
Ex. 366. Find the area of a trapezoid whose bases are, respectively, 16 ft. and 12 ft., and altitude 18 ft.

Ex. 367. Given the bases of a trapezoid, 24 ft. and 28 ft., respectively, and its altitude 14 ft., to find the altitude of an equivalent parallelogram whose base is 26 ft.

Ex. 368. Find the area of a trapezoid, having given the longer base, 85 ft., the legs, 29 ft. and 52 ft., and the altitude, 20 ft.

Ex. 369. Find the area of a trapezoid whose altitude is 18 ft. and median 26 ft.

Ex. 370. The line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equivalent parts.



Ex. 371. The areas of two triangles having an angle in the one supplementary to an angle in the other are to each other as the products of the sides including the supplementary angles.

$$\frac{\triangle ABC}{\triangle ABE}=\frac{BC}{BE} \text{ and } \frac{\triangle ABE}{\triangle BDE}=\frac{AB}{BD}.$$

Ex. 372. If two triangles have an angle in the one equal to an angle in the other and also an angle in the one supplementary to an angle in the other, the sides opposite the equal angles have the same ratio as the sides opposite the supplementary angles.

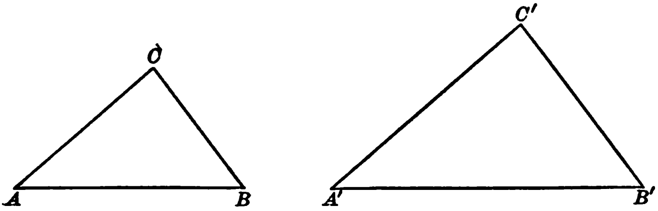
Ex. 373. The area of a rhombus is equal to half the product of its diagonals.

Ex. 374. The sum of the perpendiculars from any point within an equilateral triangle to the sides is equal to the altitude.

Ex. 375. Construct a rectangle having a given side and equivalent to a given square.

PROPOSITION VIII. THEOREM.

435. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



Given AB and A′B′, homologous sides of similar $\triangle$s ABC and A′B′C′.

To prove $$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

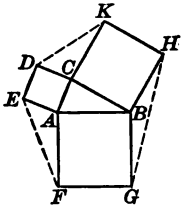
Proof. $$\measuredangle A=\measuredangle A'. \qquad \S\ 363$$

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'}=\frac{AB\times AC}{A'B'\times A'C'}=\frac{AB}{A'B'}\times\frac{AC}{A'C'}. \qquad \S\ 434$$

But $$\frac{AC}{A'C'}=\frac{AB}{A'B'}. \qquad \S\ 363$$

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'}=\frac{AB}{A'B'}\times\frac{AB}{A'B'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Q. E. D.

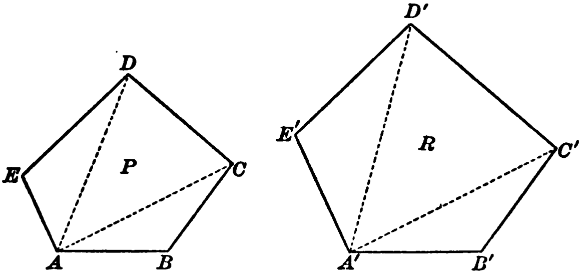


Ex. 376. If squares be constructed on the sides of any triangle and the adjacent vertices of the squares be joined, two and two, the area of each triangle thus formed is equal to the area of the original triangle.

Apply Ex. 371.

PROPOSITION IX. THEOREM.

436. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



Given two similar polygons, P and R, AB and A'B' being homologous sides.

To prove $\dfrac{P}{R}=\dfrac{\overline{AB}^2}{\overline{A'B'}^2}.$

Proof. From two homologous vertices, A and A', draw all possible diagonals. P and R are thus divided into the same number of similar triangles, similarly placed. § 376

Then $\dfrac{\triangle ABC}{\triangle A'B'C'}=\dfrac{\overline{AB}^2}{\overline{A'B'}^2},$

and $\dfrac{\triangle ACD}{\triangle A'C'D'}=\dfrac{\overline{CD}^2}{\overline{C'D'}^2}.$ § 435

But $\dfrac{\overline{CD}^2}{\overline{C'D'}^2}=\dfrac{\overline{AB}^2}{\overline{A'B'}^2}.$ §§ 338, 363

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'}=\frac{\triangle ACD}{\triangle A'C'D'}.$$

Similarly $\dfrac{\triangle ABC}{\triangle A'B'C'}=\dfrac{\triangle ADE}{\triangle A'D'E'},$ etc.

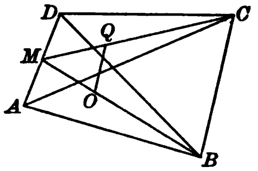
$$\therefore \frac{\triangle ABC + \triangle ACD}{\triangle A'B'C' + \triangle A'C'D'}, \text{ etc.} = \frac{\triangle ABC}{\triangle A'B'C'}. \qquad \text{§ 335}$$

$$\therefore \frac{P}{R} = \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad \text{Q. E. D.}$$

437. Cor. 1. *The areas of two similar polygons are to each other as the squares of any two homologous lines.*

438. Cor. 2. *Two homologous lines of two similar polygons are to each other as the square roots of their areas.*

Ex. 377. Divide a given trapezoid into two equivalent parts by a line parallel to the bases.



Ex. 378. *ABCD* is a quadrilateral. The centroids of the triangles, *ABC*, *ABD*, *ACD*, and *DBC* are vertices of a quadrilateral similar to *ABCD*.

Let O be the centroid of $\triangle ABD$, and Q the centroid of $\triangle ACD$. $MQ = \frac{1}{3} CM$, and $MO = \frac{1}{3} BM$ (§ 205). Therefore $OQ = \frac{1}{3} BC$, and $\parallel BC$ (§ 344). In like manner each line joining two centroids is $\parallel$ and $= \frac{1}{3}$ a side of the quadrilateral.

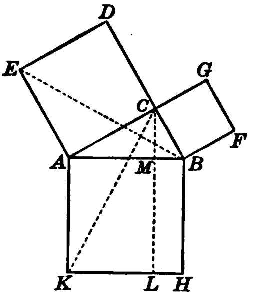
Ex. 379. Divide a given triangle into two equivalent parts by a line through one of its vertices.

Ex. 380. Divide a given triangle into two equivalent parts by a line through a given point in one of its sides.

Ex. 381. Divide a given triangle into two equivalent parts by a line parallel to one of its sides.

PROPOSITION X. THEOREM.

439. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs.*



Given AH, AD, and CF, squares on the hypotenuse AB, and the legs AC and BC of the rt. △ ABC.

To prove $\text{square } AH = \text{square } AD + \text{square } CF$.

Proof. Draw $CL \perp KH$ and draw CK and BE. § 114

Since $\angle$s BCA, ACD, and BCG are rt. $\angle$s, ACG and BCD are straight lines.

In △s BAE and CAK, $AB = AK$ and $AE = AC$, § 165

and $\angle BAE = \angle CAK$ (each = rt. $\angle + \angle BAC$).

$\therefore \triangle BAE \cong \triangle CAK$. § 91

Because $\triangle BAE$ and square AD have the same base and altitude, $\triangle BAE = \frac{1}{2}$ square AD. § 426

And because $\triangle ACK$ and rectangle AL have the same base and altitude, $\triangle CAK = \frac{1}{2}$ rectangle AL. § 426

But $\triangle BAE \cong \triangle CAK$. just proved

$\therefore$ square $AD =$ rectangle AL.

In like manner it is proved that

square $CF =$ rectangle BL.

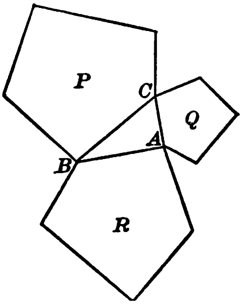
But square AH = rectangle AL + rectangle BL.

$\therefore$ square AH = square AD + square CF. Q. E. D.

NOTE. The foregoing demonstration is essentially that given by Euclid (300 B.C.). The discovery of the proposition is attributed to Pythagoras (550 B.C.), and it is known as the "Pythagorean Theorem."

PROPOSITION XI. THEOREM.

440. *If similar polygons be constructed upon the sides of a right triangle as homologous sides, the polygon on the hypotenuse is equivalent to the sum of the polygons on the two legs.*



Given similar polygons, P, Q, and R, constructed on the sides of the rt. △ ABC, BC being the hypotenuse.

To prove $P = Q + R$.

Proof. $$\frac{R}{P} = \frac{\overline{AB}^2}{\overline{BC}^2}, \qquad (1)$$

and $$\frac{Q}{P} = \frac{\overline{AC}^2}{\overline{BC}^2}. \qquad (2) \quad § 436$$

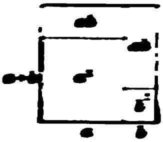
Adding (1) and (2),

$$\frac{R+Q}{P} = \frac{\overline{AB}^2 + \overline{AC}^2}{\overline{BC}^2}.$$

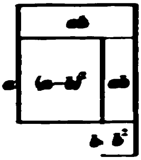
But $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$. § 439

$\therefore P = R + Q$. Q. E. D.

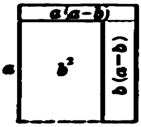
Ex. 262. The square on the sum of two lines is equal to the sum of the squares on the lines plus twice the rectangle of the lines.



Ex. 263. The square on the difference of two lines is equal to the sum of the squares on the lines minus twice the rectangle of the lines. (Correlative of Ex. 262.)



Ex. 264. The rectangle of the sum and the difference of two lines is equal to the difference of the squares on the lines.



THE PRINCIPLE OF HOMOLOGY.

441. The relation between the concepts of geometry and those of algebra is called the Principle of Homology.

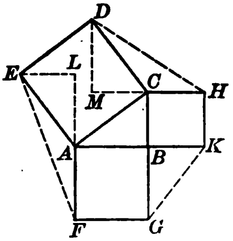
ALGEBRAIC QUANTITIES.	GEOMETRIC QUANTITIES.
a is a number.	a is a line.
ab is the product of two numbers.	ab is the rectangle of two lines.
a^2 is the second power of the number a.	a^2 is the square whose side is the line a.

etc.

ALGEBRAIC THEOREMS.	GEOMETRIC THEOREMS.
The square of the sum of two numbers is equal to the sum of their squares plus twice their product.	The square on the sum of two lines is equal to the sum of the squares on the lines plus twice their rectangle.
The product of the sum and the difference of two numbers is equal to the difference of their squares.	The rectangle of the sum and the difference of two lines is equal to the difference of their squares.

etc.

Many theorems, far from obvious in geometry, become easily so when the homologous theorems in algebra are considered.



Ex. 385. The square on the hypotenuse of a right triangle is equal to four times the area of the triangle plus the square on the difference of the legs. Prove algebraically.

Ex. 386. If the adjacent vertices of the sides of the squares on the sides of a right triangle are joined, two and two, the sum of the squares of the three lines joining them is equal to six times the square of the hypotenuse.

Ex. 387. Divide a given triangle into two equivalent parts by a line perpendicular to one of its sides.

Ex. 388. Find the area of a rhombus whose diagonals are 16 and 18.

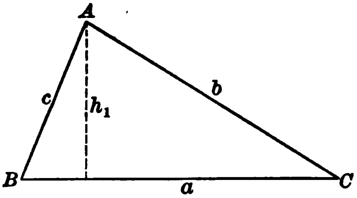
Ex. 389. Given the side of an equilateral triangle $= a$. Show that the altitude $= \frac{a}{2}\sqrt{3}$.

Ex. 390. Two similar polygons have perimeters 18 and 21 ft., respectively. If the area of the first is 480 sq. ft., what is the area of the second ?

Ex. 391. Divide a given triangle into two parts having a given ratio by a line through one of the vertices.

PROPOSITION XII. PROBLEM.

442. *To compute the area of a triangle in terms of its sides.*



Given the triangle whose sides are a, b, c, to find the area.

Solution. Denote altitude to side a by h_1.

$$\text{Area } \triangle ABC = \tfrac{1}{2} a h_1. \qquad \S\ 425$$

$$h_1 = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}. \qquad \S\ 400$$

Substituting the value of h_1,

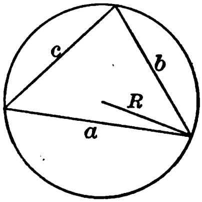
$$\text{area } \triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}. \qquad \text{Q. E. F.}$$

FORMULA: $$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

NOTE. This formula for the area of the triangle is called Hero's formula, because Hero of Alexandria was first to develop it. Hero lived about 125 B.C.

PROPOSITION XIII. PROBLEM.

443. *To compute the radius of the circumscribed circle in terms of the sides of the triangle.*



Denote the radius of the circumscribed ⊙ by R, the area of the △ by △, and the sides by a, b, c.

$$2\,Rh_1 = bc. \qquad \S\ 407$$

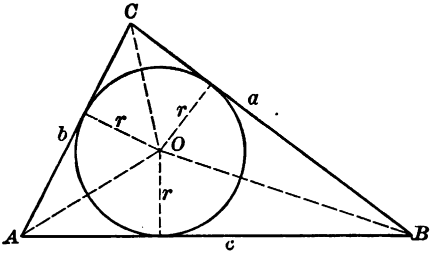
$$2\,aRh_1 = abc.$$

But $$ah_1 = 2\,\triangle. \qquad \S\ 425$$

Hence $$R = \frac{abc}{4\,\triangle}. \qquad \text{Q. E. F}$$

PROPOSITION XIV. PROBLEM.

444. *To compute the radius of the inscribed circle in terms of the sides of the triangle.*



Denote the radius of the inscribed circle by r, the sides by a, b, c, the perimeter by $2\,s$, and the area by △.

Then $$\triangle\,AOB = \tfrac{1}{2}\,cr,$$

$$\triangle\,AOC = \tfrac{1}{2}\,br,$$

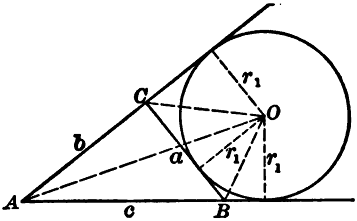
and $$\triangle\,BOC = \tfrac{1}{2}\,ar. \qquad \S\ 425$$

$$\therefore \triangle\,ABC = \tfrac{1}{2}\,r(a+b+c) = \tfrac{1}{2}\,r \times 2\,s = rs.$$

Whence $$r = \frac{\triangle}{s} \qquad \text{Q. E. F.}$$

PROPOSITION XV. PROBLEM.

445. *To compute the radii of the escribed circles in terms of the sides of the triangle.*



Denote the radii of the escribed circles of the triangle by r_1 r_2, r_3, the sides by a, b, c, and the area by Δ.

$$\triangle AOB = \tfrac{1}{2} cr_1,$$

$$\triangle AOC = \tfrac{1}{2} br_1,$$

and $$\triangle BOC = \tfrac{1}{2} ar_1. \qquad \S\ 425$$

But $$\triangle ABC = \triangle AOB + \triangle AOC - \triangle BOC.$$

$$\therefore \Delta = \tfrac{1}{2} cr_1 + \tfrac{1}{2} br_1 - \tfrac{1}{2} ar_1$$

$$= \tfrac{1}{2} r_1(b + c - a)$$

$$= r_1(s - a).$$

Whence $$r_1 = \frac{\Delta}{s-a}.$$

Similarly $$r_2 = \frac{\Delta}{s-b}, \text{ and } r_3 = \frac{\Delta}{s-c}.$$ Q. E. F.

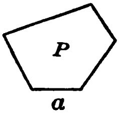
Ex. 392. Given the radius of the inscribed circle, r, and the radii of the escribed circles, r_1, r_2, r_3, of a triangle, to prove $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

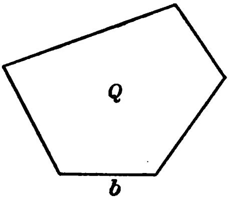
Ex. 393. Prove $\Delta = \sqrt{r\, r_1\, r_2\, r_3}$.

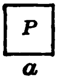
Ex. 394. Prove $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

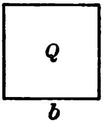
Proposition XVI. Problem.

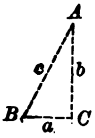
446. *To construct a polygon similar to two given similar polygons and equivalent to their sum.*

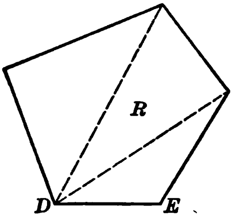


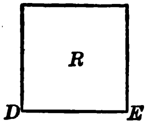












Given two similar polygons, P and Q, to construct a polygon R, similar to P and Q and equal to P + Q.

Construction. Let a and b be two homologous sides of P and Q.

Construct a rt. △ ABC, having leg $BC = a$ and leg $AC = b$. § 302

Draw $DE = AB$ and on DE as side homologous to a, construct polygon R similar to P. § 378

Then R is the polygon required.

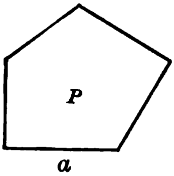
Proof. Because R is similar to P by construction, it is similar to Q, § 373

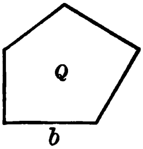
and $R = P + Q$. § 440

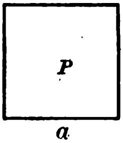
Q. E. F.

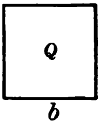
Proposition XVII. Problem.

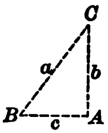
447. *To construct a polygon similar to two given similar polygons and equivalent to their difference.*

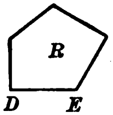


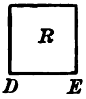












Given two similar polygons, P and Q, to construct a polygon R, similar to P and Q, and equal to P − Q.

Construction. Let a, b, be two homologous sides of P and Q.

Construct a rt. △ ABC having hypotenuse $BC = a$ and leg $AC = b$.

Draw $DE = AB$ and on DE as a side homologous to a, construct polygon R similar to P. § 378

Then R is the polygon required.

Proof. Because R is similar to P by construction, it is similar to Q, § 373

and $R = P - Q$. § 440

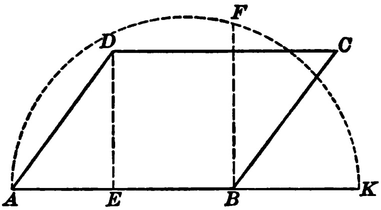
Q.E.F.

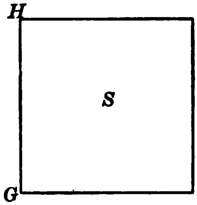
Ex. 395. Construct a square equivalent to the sum of four given squares.

Ex. 396. Find the area of the triangle whose sides are 13, 14, and 15 ft., respectively.

PROPOSITION XVIII. PROBLEM.

448. *To construct a square equivalent to a given parallelogram.*





Given the ▱ ABCD.

Required to construct a square = ▱ ABCD.

Construction. Draw *DE*, the altitude of ▱ *ABCD*. § 114

Construct *BF*, the mean proportional between *AB* and *DE*. § 386

Construct the square *S*, whose side $GH = BF$. Then *S* is the square required.

Proof. $S = \overline{GH}^2 = \overline{BF}^2 = AB \times DE$ Cons.

But ▱ $ABCD = AB \times DE$. § 420

$\therefore S =$ ▱ $ABCD$. Q. E. F.

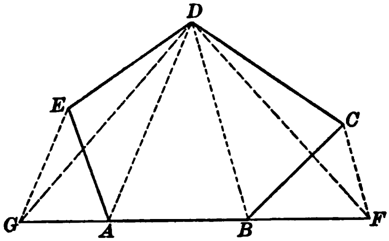
449. Cor. ***A square may be constructed equivalent to a given triangle by taking for its side the mean proportional between the base and half the altitude of the triangle.***

Ex. 397. Find the area of the triangle whose sides are, respectively, 10, 17, and 21 ft.

Ex. 398. Find the length of the longest chord and the shortest chord that can be drawn through a point 8 in. from the center of a circle whose radius is 17 in.

Proposition XIX. Problem.

450. *To construct a triangle equivalent to a given polygon.*



Given the polygon ABCDE.

Required to construct a triangle equivalent to $ABCDE$.

Construction. Let B, C, and D be three consecutive vertices.

Draw the diagonal BD and draw $CF \parallel BD$ to meet AB produced at F. § 154

Draw DF.

In like manner reduce the number of sides of the polygon until the triangle DFG is obtained.

Then DFG is the $\triangle$ required.

Proof. The polygon $AFDE$ has one side less than $ABCDE$, but they have equal areas, for the part $ABDE$ is common.

$$\triangle BFD = \triangle BCD, \qquad \text{§ 430}$$

(for they have the same base BD and the same altitude, the perpendicular distance between the ‖s *CF and BD).*

In like manner it is proved that $\triangle DFG = AFDE$, for the part ADF is common and $\triangle ADE = \triangle ADG$.

$$\therefore \triangle DFG = \text{polygon } ABCDE. \qquad \text{Q. E. F.}$$

451. Cor. *By applying § 450 and § 449, a square may be constructed equivalent to any given polygon.*

Ex. 399. Given the sides of a triangle, 7, 15, and 20 ft., to find the radius of the circumscribed circle.

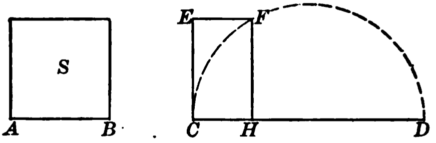
Ex. 400. Given the sides of a triangle, 12, 55, and 65 ft., to find the area.

Ex. 401. Given the area of a triangle, 42 sq. ft., the radius of the inscribed circle 2 ft., and the radius of one escribed circle 3 ft. Find the sides of the triangle.

Ex. 402. Given the radii of the three escribed circles of a triangle, 2, 8, and 24 ft., respectively. Find the area of the triangle.

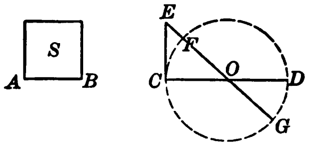
Ex. 403. Given the perimeter of a triangle, 84 ft., one altitude, $9\frac{3}{5}$ ft., and the area, 168 sq. ft. Find the radius of the circumscribed circle.

Ex. 404. Given the altitudes of a triangle, 21, 28, and 60 ft. Find the sides of the triangle.



Ex. 405. Construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.

Let AB be side of given square, S, and CD the given line. Describe semicircle on CD. Draw $CE \perp CD$ and $= AB$. Draw $EF \parallel CD$ and $FH \perp CD$. Rectangle DH by $CH = S$.

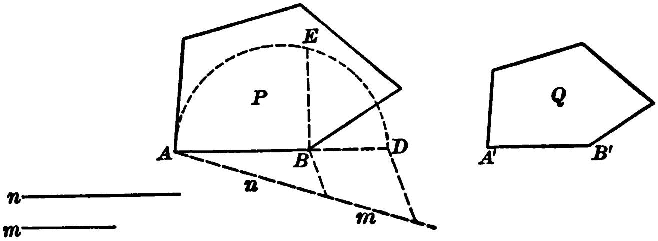


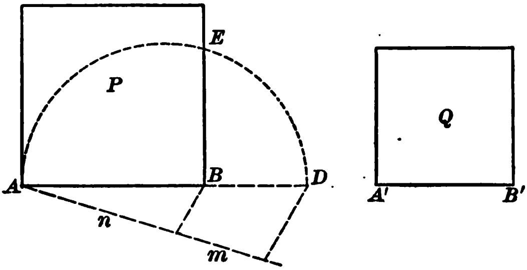
Ex. 406. Construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.

Let AB be the side of the given square, S, and CD the given line. Draw $CE \perp CD$ and AB. Draw EG through center O. Rectangle EG by $EF = S$.

Proposition XX. Problem.

452. *To construct a polygon similar to a given polygon and having a given ratio to it.*





Given the polygon P, and the lines m and n.

Required to construct a polygon Q similar to P, such that

$$\frac{Q}{P}=\frac{m}{n}.$$

Construction. Produce AB, one side of P, to D, making

$$\frac{BD}{AB}=\frac{m}{n}. \qquad \text{§ 387}$$

Construct a mean proportional, BE, between AB and BD. § 386

On $A'B'=BE$ as side homologous to AB construct polygon Q similar to P. § 378

Then Q is the polygon required.

Proof. $$\frac{AB}{A'B'} = \frac{A'B'}{BD}.$$ Cons.

$$\therefore \overline{A'B'}^2 = AB \times BD.$$

But $$\frac{Q}{P} = \frac{\overline{A'B'}^2}{\overline{AB}^2}.$$ § 436

$$\therefore \frac{Q}{P} = \frac{AB \times BD}{AB \times AB} = \frac{BD}{AB} = \frac{m}{n}.$$ Q. E. F.

Ex. 407. To divide a given trapezoid into two similar trapezoids by a line parallel to the bases.

Ex. 408. The area of the mid-point parallelogram of a quadrilateral (Ex. 43) is half the area of the quadrilateral.

Ex. 409. The area of a trapezoid is equal to the product of one leg by the line drawn perpendicular to this leg from the mid-point of the other leg.

Ex. 410. The sides of a triangle are 4 ft., 13 ft., and 15 ft. What are the sides of a similar triangle whose area is 96 sq. ft.?

Ex. 411. Given the radii of the three escribed circles of a triangle, 21, 24, and 28 ft., respectively. Find the area of the triangle.

Ex. 412. Given the area of a triangle, 84 sq. yd., one side 13 yd., and the radius of the inscribed circle, 4 yd. Find the radius of the circumscribed circle.

Ex. 413. Given the altitudes of a triangle, 6, 8, and 12 ft., respectively, to find the radius of the inscribed circle.

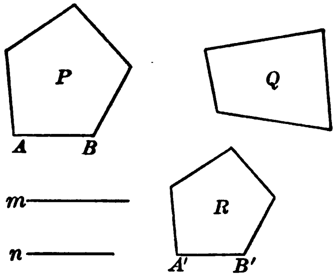
Ex. 414. Which of two triangles has the greater area, the one whose sides are, respectively, 25, 51, and 74 ft., or the one whose sides are 20, 37, and 51 ft.?

Ex. 415. Construct a triangle having a given base and equivalent to the sum of two given triangles.

Ex. 416. Construct a triangle having a given altitude, and equivalent to the difference of two given rectangles.

PROPOSITION XXI. PROBLEM.

453. *To construct a polygon similar to one of two given polygons and equivalent to the other.*



Given two polygons, P and Q.

Required to construct a polygon $R \sim P$ *and* $= Q$.

Construction. Construct m, the side of a square equivalent to P, and n, the side of a square equivalent to Q. § 451

Construct $A'B'$, the fourth proportional to m, n, and AB, one side of P. § 387

On $A'B'$ as side homologous to AB construct the polygon R similar to P. § 378

Then R is the polygon required.

Proof. $\frac{m}{n} = \frac{AB}{A'B'}$. Cons.

$\therefore \frac{\overline{m}^2}{\overline{n}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$. § 338

But $P = m^2$ and $Q = n^2$. Cons.

$\therefore \frac{P}{Q} = \frac{\overline{m}^2}{\overline{n}^2}$.

$\therefore \frac{P}{Q} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$

But $\frac{P}{R} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$. § 436

$$\therefore \frac{P}{Q} = \frac{P}{R}.$$

Hence $R = Q$. § 326

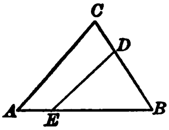
Q. E. F.

Ex. 417. Construct an equilateral triangle equivalent to a given square.

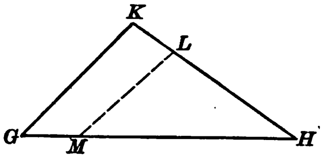
Ex. 418. Construct a square equivalent to $\frac{5}{6}$ a given square.

Ex. 419. Construct a square equivalent to $\frac{2}{3}$ a given triangle.

Ex. 420. Construct a square equivalent to half a given pentagon.



Ex. 421. Given the triangle ABC. If on BC is laid off BD equal to $\frac{2}{3}$ BC, and on BA is laid off BE equal to $\frac{3}{4}$ BA, what part of the triangle ABC is the triangle BED?



Ex. 422. Given the triangle GHK. On HK is taken HL equal to $\frac{5}{6}$ HK, and from L a line is drawn to meet HG at M, making the triangle HLM equal to $\frac{7}{10}$ the triangle GHK. What part of HG is the line HM?

Ex. 423. Find the point within a triangle from which lines to the three vertices divide the triangle into three equivalent parts.

Ex. 424. Having given the polygons, P, Q, and R, construct a polygon S equal to the sum of P and Q and similar to R.

Ex. 425. What is the ratio of the areas of two equilateral triangles having the side of the first equal to the altitude of the second?

Book V.

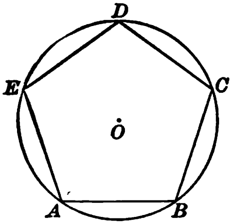
REGULAR POLYGONS.

MEASUREMENT OF THE CIRCLE.

454. Def. A **regular polygon** is a polygon which is both equilateral and equiangular.

Proposition I. Theorem.

455. *An equilateral polygon inscribed in a circle is regular.*



Given the equilateral polygon ABCDE inscribed in the circle O.

To prove $ABCDE$ *is a regular polygon.*

Proof. The arcs AB, BC, CD, etc. are all equal. § 234

$\therefore$ arc $BCDE =$ arc $CDEA$,

(*each being equal to the circle less two of the equal arcs*).

$\therefore \measuredangle A = \measuredangle B$. § 289

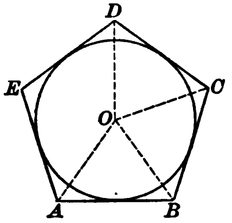
Similarly, $\measuredangle B = \measuredangle C$, etc.

$\therefore$ the polygon is equiangular, and therefore regular.

Q. E. D

PROPOSITION II. THEOREM.

456. *An equiangular polygon circumscribed about a circle is regular* (correlative of § 455).



Given the equiangular polygon ABCDE circumscribed about the circle O.

To prove *ABCDE is a regular polygon.*

Proof. Draw OA, OB, OC, and OD.

Each of these lines is the bisector of an angle of the polygon. § 258

In △s AOB and BOC,

$$OB = OB. \qquad \text{Iden.}$$

$$\angle OAB = \angle OBA = \angle OBC = \angle OCB,$$

(*each being half an $\angle$ of an equiangular polygon*).

$$\therefore \triangle AOB \cong \triangle BOC. \qquad \text{§§ 159, 94}$$

$$\therefore AB = BC. \qquad \text{§ 85}$$

Similarly, $BC = CD = DE$, etc.

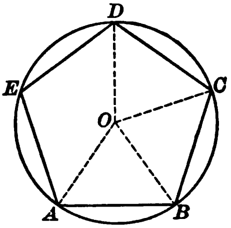
∴ the polygon $ABCDE$ is regular, being both equiangular and equilateral. Q. E. D.

Ex. 426. An equiangular polygon inscribed in a circle is regular if it has an odd number of sides.

Ex. 427. An equilateral polygon circumscribed about a circle is regular if it has an odd number of sides. (Correlative of Ex. 426.)

PROPOSITION III. THEOREM.

457. *A circle may be circumscribed about or inscribed in a regular polygon.*



Given the regular polygon ABCDE.

To prove 1. *A circle may be circumscribed about ABCDE.*

2. *A circle may be inscribed in ABCDE.*

Proof. 1. Let O be the center of the ⊙ described through the three points A, B, and C. § 246

Draw OA, OB, OC, and OD.

$\angle ABC = \angle BCD$, (1) § 454

and $OC = OB$. § 213

$\therefore \angle OBC = \angle OCB$. (2) § 98

Subtracting (2) from (1),

$\angle OBA = \angle OCD$.

Also $AB = CD$. § 454

$\therefore \triangle OAB \cong \triangle OCD$. § 91

$\therefore OD = OA$. § 85

Hence the ⊙ through A, B, and C passes through D. In like manner it is proved to pass through E, etc.

$\therefore$ A ⊙ may be circumscribed about the polygon $ABCDE$.

2. Since the sides of $ABCDE$ are all equal chords of the circumscribed ⊙, they are equidistant from the center O. § 241

$\therefore$ with O as a center and a radius equal to the perpendicular distance to any side of the polygon a ⊙ may be inscribed.

Q. E. D.

458. Def. The **center** of a regular polygon is the common center of its inscribed and circumscribed circles.

459. Def. The **radius** of a regular polygon is the radius of its circumscribed circle.

460. Def. The **apothem** of a regular polygon is the radius of its inscribed circle.

461. Def. The **central angle** of a regular polygon is the angle made by two radii drawn to the extremities of the same side.

462. Cor. 1. *The central angles of a regular polygon are all equal, each being equal to a perigon divided by the number of sides.*

463. Cor. 2. *The central angle of a regular polygon is the supplement of an interior angle.*

464. Cor. 3. *The radius of a regular polygon bisects an interior angle.*

Ex. 428. What is the value in degrees of the central angle of a regular polygon of 20 sides?

Ex. 429. What is the value in degrees of an interior angle of a regular polygon of 18 sides?

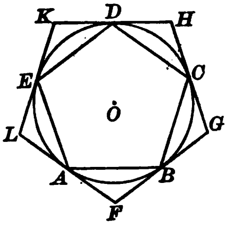
Ex. 430. How many sides has the regular polygon in which an interior angle is 9 times a central angle?

Proposition IV. Theorem.

465. *If a circle be divided into any number of equal arcs,*

1. *The chords joining the points of division form a regular inscribed polygon.*

2. *The tangents at the points of division form a regular circumscribed polygon.*



Given the ⊙ O, divided into equal arcs at the points A, B, C, D, etc.; also the tangents FG, GH, HK, etc., at the points A, B, C, etc.

To prove 1. *ABCDE is a regular polygon.*

2. *FGHKL is a regular polygon.*

Proof. 1. Chords AB, BC, CD, etc., are all equal. § 233

∴ $ABCDE$ is equilateral.

∴ $ABCDE$ is a regular polygon. § 455

2. Since arcs AB, BC, CD, etc. are all equal by hypothesis,

$\measuredangle BAF = \measuredangle FBA = \measuredangle GBC = \measuredangle GCB$, etc. § 292

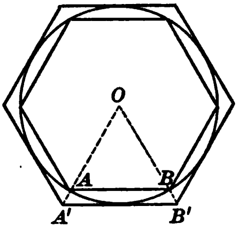
∴ △s BAF, BGC, CHD, etc., are all congruent isosceles △s. § 94

∴ $\measuredangle F = \measuredangle G = \measuredangle H = \measuredangle K$, etc. § 85

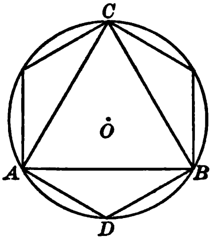
∴ $FGHKL$ is a regular polygon. § 456

Q. E. D.

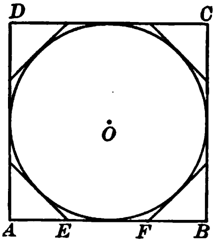
466. Cor. 1. *Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides as the inscribed polygon.*



467. Cor. 2. *Tangents to a circle at the mid-points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon whose sides are parallel to the sides of the inscribed polygon and whose vertices are on the prolongations of the radii of the inscribed polygon.*



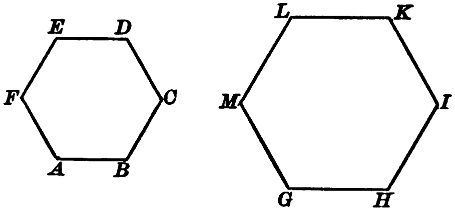
468. Cor. 3. *If at the mid-point of each arc subtended by a side of a regular inscribed polygon lines be drawn to its extremities, a regular inscribed polygon of twice the number of sides is formed.*



469. Cor. 4. *Tangents at the mid-points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of twice the number of sides.*

Proposition V. Theorem.

470. *Regular polygons of the same number of sides are similar.*



Given the regular polygons ABCDEF and GHIKLM, of the same number of sides.

To prove $ABCDEF \sim GHIKLM$.

Proof. $AB = BC = CD$, etc., and $GH = HI = IK$, etc. § 454

$$\therefore \frac{AB}{GH} = \frac{BC}{HI} = \frac{CD}{IK}, \text{ etc.} \qquad \text{Ax. 5}$$

∴ The polygons have their homologous sides proportional.

Again, each angle of the two polygons $= \frac{n-2}{n}$ st. $\angle$s (§ 193), and are, therefore, equal.

$$\therefore ABCDEF \sim GHIKLM. \qquad § 363$$

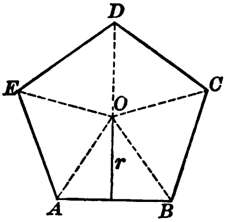
Q. E. D.

471. Cor. 1. *The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.*

472. Cor. 2. *The areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous lines.*

PROPOSITION VI. THEOREM.

473. *The area of a regular polygon is half the product of its perimeter and apothem.*



Given the regular polygon ABCDE, whose perimeter is P and apothem is r.

To prove area $ABCDE = \frac{1}{2} P \times r$.

Proof. From the center O draw radii OA, OB, OC, etc.

The △s thus formed have the same altitude r. § 457

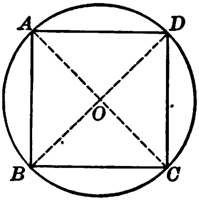
∴ the area of each $\triangle = \frac{1}{2} r \times$ by a side of the polygon. § 425

∴ the sum of the areas of all the $\triangle\text{s} = \frac{1}{2} r \times$ by the sum of all the sides of the polygon,

or area $ABCDE = \frac{1}{2} P \times r$. Q. E. D.

PROPOSITION VII. PROBLEM.

474. *To inscribe a square in a given circle.*



Given the circle whose center is O.

Required *to inscribe a square in circle O.*

Construction. Draw two diameters, AC and BD $\perp$ each other. § 113

Draw AB, BC, CD, and DA.

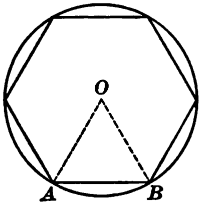
Then $ABCD$ is the square required.

Proof. Each angle of the polygon $ABCD$ is a right angle (§ 290) and the sides are all equal. § 233

$\therefore ABCD$ is a square. Q. E. F.

PROPOSITION VIII. PROBLEM.

475. *To inscribe a regular hexagon in a given circle.*



Given the circle O.

Required *to inscribe a regular hexagon in circle O.*

Construction. Draw any radius, as OA.

With A as a center and radius equal to OA describe an arc intersecting the ⊙ at B.

Draw AB.

Then AB is the side of the hexagon required.

Hence, to inscribe a regular hexagon, apply the radius to the ⊙ six times as a chord.

Proof. Draw OB.

The $\triangle AOB$ is equilateral by construction.

$\therefore \measuredangle AOB = \frac{2}{3}$ rt. $\measuredangle$ or $\frac{1}{6}$ a perigon. § 162

$\therefore$ the arc AB is $\frac{1}{6}$ the ⊙.

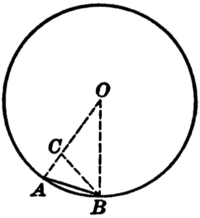
Hence the chord AB is the side of a regular inscribed hexagon. Q. E. F.

476. Cor. 1. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

477. Cor. 2. *The chords joining the alternate vertices of a regular inscribed hexagon form an inscribed equilateral triangle.*

Proposition IX. Problem.

478. *To inscribe a regular decagon in a given circle.*



Given the circle O.

Required to inscribe a regular decagon in ⊙ *O.*

Construction. Draw any radius, as OA, and divide it internally in extreme and mean ratio, OC being the greater segment. § 395

With A as a center and radius equal to OC describe an arc intersecting the circle at B.

Draw AB.

Then AB is the side of the decagon required.

Hence, to inscribe a regular decagon apply the greater segment of the radius divided internally in extreme and mean ratio to the circle ten times as a chord.

Proof. Draw OB and BC.

In △s AOB and ABC,

$\measuredangle A = \measuredangle A$, Iden.

and $OA : AB = AB : AC.$ Cons.

$\therefore \triangle AOB \sim \triangle ABC$, § 368

(having an ∡ in each equal and the including sides proportional).

$\therefore \angle ABC = \angle AOB.$ (1) § 85

$\triangle AOB$ is isosceles. § 213

$\therefore \triangle ABC$ is isosceles,

and $BC = AB = OC.$

$\therefore \angle OBC = \angle AOB.$ (2) § 85

Adding (1) and (2),

$\angle ABO = 2 \angle AOB.$

But $\angle BAO = \angle ABO.$ § 98

$\therefore \angle ABO + \angle BAO = 4 \angle AOB.$

But $\angle ABO + \angle BAO =$ st. $\angle - \angle AOB.$ § 155

$\therefore 4 \angle AOB =$ st. $\angle - \angle AOB.$

Hence, $5 \angle AOB =$ st. $\angle$.

$\therefore \angle AOB = \frac{1}{5}$ st. $\angle = \frac{1}{10}$ perigon.

$\therefore$ arc AB is $\frac{1}{10}$ the circle, and the chord AB is a side of a regular inscribed decagon. Q. E. F.

479. Cor. 1. *The side of a regular inscribed decagon is equal to the greater segment of the radius divided internally in extreme and mean ratio.*

480. Cor. 2. *The chords joining the alternate vertices of a regular inscribed decagon form a regular inscribed pentagon.*

Ex. 431. In the triangle ABC, $\angle B$ is 3 times $\angle A$ and $\angle C$ is 4 times $\angle A$. Find the number of degrees in $\angle A$.

Ex. 432. How many sides has a polygon, the sum of whose interior angles is 900°?

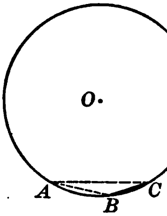
Ex. 433. The diameter of a given circle is 106 ft. How far from the center is a chord whose length is 90 ft.?

Ex. 434. Two secants intercept arcs of 60° and 84° on a circle. How many degrees in the angle formed by the secants?

Ex. 435. Two chords intersecting within a circle form an angle of 60°. If one of the intercepted arcs is 90°, how many degrees in the other arc?

PROPOSITION X. PROBLEM.

481. *To inscribe a regular pentedecagon in a given circle.*



Given the circle O.

Required to inscribe a regular pentedecagon in ⊙ O.

Construction. Construct the chord AC equal to the side of a regular inscribed hexagon (§ 475), and the chord AB equal to the side of a regular inscribed decagon (§ 478).

Then BC is the side of the pentedecagon required.

Hence, to inscribe a regular pentedecagon apply the chord BC to the circle fifteen times.

Proof. The arc AC is $\frac{1}{6}$ the circle, Cons.

and the arc AB is $\frac{1}{10}$ the circle. Cons.

$\therefore$ arc BC is $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ the circle. Hence chord BC is the side of a regular inscribed pentedecagon. Q. E. F.

482. NOTE. By bisecting the arcs whose chords are the sides of an inscribed square, we may construct a regular inscribed octagon (§ 468), and by successive bisections, we may construct regular inscribed polygons of 16, 32, 64, etc., sides.

Similarly, by successive bisections of the arcs whose chords are sides of a regular inscribed hexagon, decagon, and pentedecagon, respectively, we may construct regular inscribed polygons of 12, 24, 48, etc., of 20, 40, 80, etc., and of 30, 60, 120, etc., sides.

The inscription and circumscription and, therefore, the construction of regular polygons are seen to depend upon the division of the circle or of the perigon. In the construction of regular polygons, elementary geometry is therefore limited to the construction of regular polygons of 3×2^n, 4×2^n, 5×2^n, and 15×2^n sides, or, as it may be expressed,

$$\left.\begin{matrix}3\\4\\5\\15\end{matrix}\right\} \times 2^n$$

sides, n being positive and integral.

The German mathematician, Gauss (1777–1855), has proved it possible to inscribe with the ruler and compass, the only instruments of elementary geometry, a regular polygon of 17 sides, and further, that a circle or a perigon may be divided into $2^n + 1$ equal parts, when n is a positive integer, and $2^n + 1$ is a prime number; but the method of Gauss is too complicated for the present work.

483. Note. By § 193 it is found that the interior angle of a regular polygon of 3 sides is 60°; of 4 sides, 90°; of 5 sides, 108°; and of 15 sides, 156°. The central angles of these polygons are also found to be, respectively, 120°, 90°, 72°, and 24°.

It is seen that each of these angles is a multiple of 3°; therefore the sum of any number of these angles, or the difference between any two of them, is a multiple of 3°.

Now, since any angle may be bisected by the use of the ruler and compass (§ 129), it is seen (disregarding the problem of Gauss) that any angle capable of construction with the ruler and compass that is exactly expressible in degrees, may be expressed by $\frac{3m}{2n}^\circ$, m and n being any positive integral numbers.

Denoting the radius of the circle by R.

Ex. 436. The side of an inscribed regular decagon is $\frac{R}{2}(\sqrt{5} - 1)$.

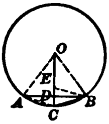
Ex. 437. The side of an inscribed regular hexagon is R.

Ex. 438. If the side of an equilateral triangle is a, the area is $\frac{a^2\sqrt{3}}{4}$.

Ex. 439. The sides of a triangle are 10, 15, and 20. Find the segments of the longest side made by the bisector of the opposite angle.

Ex. 440. Two triangles, ABC and $A'B'C'$, are similar. The area of ABC is 64 sq. ft. and $A'B'C'$ 256 sq. ft. If an altitude of triangle ABC is 8 ft., what is the length of the homologous altitude of triangle $A'B'C'$?

Ex. 441. The square on the side of a regular inscribed pentagon is equal to the square on the side of a regular inscribed hexagon plus the square on the side of a regular inscribed decagon.



Let AB be the side of the inscribed regular pentagon and BC the side of the inscribed regular decagon. Lay off $OE = BC$. Then $\triangle BEO$ is isosceles (§ 479). Also $\triangle BEC$ is isosceles.

Since BD is $\perp$ EC, $ED = CD$. § 118

Then $$CD = \frac{OC - OE}{2}.$$

Denoting the side of the pentagon by P, side of the hexagon by H, and side of the decagon by D,

$$\overline{BD}^2 = \overline{BC}^2 - \overline{CD}^2, \quad \text{§ 392}$$

or $$(\tfrac{1}{2}P)^2 = D^2 - \left(\frac{H-D}{2}\right)^2.$$

That is, $$\tfrac{1}{4}P^2 = D^2 - \frac{H^2 - 2\,DH + D^2}{4},$$

or $$P^2 = 3\,D^2 + 2\,DH - H^2.$$

But $$H : D = D : H - D. \quad \text{§§ 476, 479}$$

Whence $$D^2 = H^2 - DH,$$

or $$DH = H^2 - D^2.$$

$$\therefore P^2 = 3\,D^2 + 2\,H^2 - 2\,D^2 - H^2,$$

or $$P^2 = H^2 + D^2.$$

Denoting the radius of the circle by R,

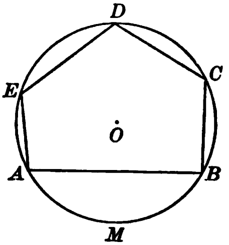
Ex. 442. The side of an inscribed equilateral triangle is $R\sqrt{3}$.

Ex. 443. The side of an inscribed square is $R\sqrt{2}$.

Ex. 444. The side of an inscribed regular pentagon is $\frac{R}{2}\sqrt{10 - 2\sqrt{5}}$.

PROPOSITION XI. THEOREM.

484. *The circle is greater than the perimeter of an inscribed polygon.*



Given the circle O and the inscribed polygon ABCDE.

To prove the $\odot$ $O >$ perimeter of $ABCDE$.

Proof. Arc $AMB >$ chord AB. § 10 (*a*)

Similarly, each arc is greater than the subtending chord.

$\therefore$ The sum of all the arcs $>$ the sum of all the chords, or the $\odot$ $O >$ perimeter of $ABCDE$.

Q. E. D.

Ex. 445. If the ratio of two homologous sides of two similar polygons is 1 : 3, what is the ratio of their areas?

Ex. 446. Find the hypotenuse of a right triangle whose legs are, respectively, 84 yd. and 187 yd.

Ex. 447. Find the side of a square whose diagonal is 10 yd.

Ex. 448. Find the side of a square whose area is 3969 sq. ft.

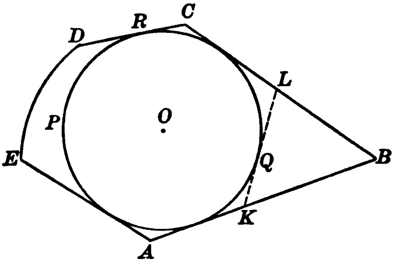
Ex. 449. In the triangle ABC, $AB = 52$ ft., $AC = 15$ ft., and $BC =$ 41 ft. Find the projections of AC and BC on AB.

Ex. 450. The external segment of a secant 16 ft. long is 9 ft. Find the length of the tangent from the same point to the same circle.

Ex. 451. Two chords, AB and CD, intersect within the circle at Q. Given $AQ = 18$, $BQ = 8$, and $CQ = 9$, to find DQ.

PROPOSITION XII. THEOREM.

485. *The circle is less than the perimeter of a circumscribed polygon or any enveloping line.*



Given the ⊙ PQR, and ABCDE, an enveloping line.

To prove ⊙ *PQR* < *perimeter of ABCDE.*

Proof. At Q draw a tangent to the circle cutting AB at K and BC at L. § 309

$$KL < KB + BL. \qquad \S\ 10\ (a)$$

∴ the enveloping line $AKLCDE$ < the line $ABCDE$.

Hence every line enveloping the ⊙ may be shortened.

∴ the ⊙ remains as the shortest line. Q. E. D.

Ex. 452. Find the area of a rectangle of which two consecutive sides are, respectively, 8 ft. and 12 ft. in length.

Ex. 453. Find the area of a parallelogram whose base is 20 and altitude 13.

Ex. 454. Find the area of a square whose diagonal is 12 mi. long.

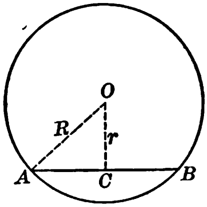
Ex. 455. Find the area of a rhombus whose diagonals are 16 yd. and 22 yd.

Ex. 456. Find the area of a trapezoid whose bases are 16 ft. and 8 ft., and whose altitude is 12 ft.

Ex. 457. Find the area of a trapezoid whose median is 12 m. and altitude 10 m.

PROPOSITION XIII. THEOREM.

486. *If the number of sides of an inscribed polygon be indefinitely increased, the apothem of the polygon will approach the radius of the circle as a limit.*



Given AB, a side, and r, the apothem of a regular polygon inscribed in a ⊙ whose radius is R.

To prove that r approaches R as a limit when the number of sides of the polygon is indefinitely increased.

Proof. $R - r < AC.$ § 86

$AB = 2\,AC.$ § 238

$\therefore R - r < \frac{1}{2} AB.$

But since AB becomes indefinitely small when the number of sides is indefinitely increased, $R - r$ approaches zero as a limit.

That is, r approaches R as a limit. § 274

Q. E. D.

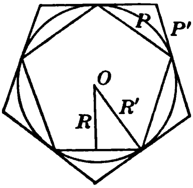
487. COR. *The square of the apothem of a regular inscribed polygon approaches the square of the radius as a limit as the number of sides of the polygon is indefinitely increased.*

Ex. 458. Find the area of a triangle whose base is 18 m. and altitude 14 m.

Ex. 459. Find the area of a triangle whose sides are, respectively, 61, 91, and 100 ft.

PROPOSITION XIV. THEOREM.

488. *The circle is the limit of the perimeters of regular inscribed and circumscribed polygons and the area of the circle is the limit of the areas of these polygons when the number of their sides is indefinitely increased.*



Given P and P', the perimeters, R and R', the apothems, S and S', the areas of the inscribed and circumscribed polygons, respectively, and C the ⊙ and K the area of the ⊙.

To prove 1. *C is the common limit of P and P'.*

2. *K is the common limit of S and S'.*

Proof. 1. $P' : P = R' : R.$ § 471

By division, $P' - P : P = R' - R : R.$ § 333

Whence $$P' - P = \frac{P(R' - R)}{R}.$$

Now $P < C.$ § 484

$$\therefore P' - P < \frac{C(R' - R)}{R}.$$

But when the number of sides of the polygon is indefinitely increased, $R' - R$ approaches zero as a limit.

Since C and R' are both constants, the limit of $\frac{C(R' - R)}{R}$ (§ 276) is zero.

But $P' > C$ and $P < C.$ §§ 485, 484

Therefore, the limit of $P' - P$ is zero.

Hence P' and P can be made to differ from each other by less than any assigned constant and each to differ from C, an intermediate quantity, by less than any assigned quantity, but not zero.

Therefore C is the common limit of P and P'.

2. Since the polygons are similar by hypothesis,

$$S' : S = R'^2 : R^2. \qquad \S\ 437$$

By division, $$S' - S : S = R'^2 - R^2 : R^2. \qquad \S\ 333$$

Whence $$S' - S = \frac{S(R'^2 - R^2)}{R^2}.$$

Now $$S < K. \qquad \text{Ax. } 7$$

Therefore, $$S' - S < \frac{K(R'^2 - R^2)}{R^2}.$$

But the limit of $R'^2 - R^2$ is zero. § 487

Therefore the limit of $\dfrac{K(R'^2 - R^2)}{R^2}$ is zero. § 276

Hence the limit of $S' - S$ is zero.

But $$S' > K \text{ and } K < S.$$

Therefore, since the difference between S' and S can be made less than any assigned quantity, the difference between K and either S' or S can be made less than any assigned quantity, but not zero.

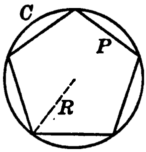
$\therefore$ K is the common limit of S and S'. Q. E. D.

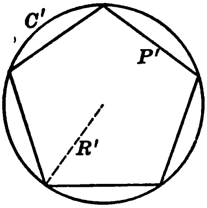
Ex. 460. Find the area of the quadrilateral $ABCD$, if $AB = 165$ yd., $BC = 85$ yd., $CD = 22$ yd., $AD = 80$ yd., and diagonal $BD = 91$ yd.

Ex. 461. In the triangle ABC, AB is 840 yd. and CD is drawn to meet AB at D so as to make the triangle BCD equal to one third the triangle ACD. Find the length of BD.

PROPOSITION XV. THEOREM.

489. *Two circles are to each other as their radii.*





Given C and C′, the $\odot$s, and R and R′, the radii, respectively, of the $\odot$s.

To prove $$\frac{C}{C'}=\frac{R}{R'}.$$

Proof. Let regular polygons of the same number of sides be inscribed in the circles and let P and P', respectively, denote their perimeters.

Then, $$\frac{P}{P'}=\frac{R}{R'}. \qquad \text{§ 471}$$

Whence $$P\times R'=P'\times R. \qquad \text{§ 324}$$

As the number of sides of the two polygons is indefinitely increased, the polygons continuing similar, P approaches C as its limit and P' approaches C' as its limit. § 488

Therefore $P\times R'$ approaches $C\times R'$ as a limit,

and $P'\times R$ approaches $C'\times R$ as a limit. § 277

$$\therefore C\times R'=C'\times R. \qquad \text{§ 275}$$

Whence $$\frac{C}{C'}=\frac{R}{R'}. \qquad \text{§ 328}$$

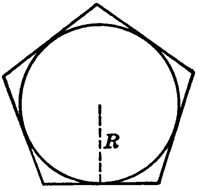
Q. E. D.

490. COR. 1. *Two circles are to each other as their diameters.*

491. COR. 2. *The constant ratio of the circle to the diameter being denoted by* π, $C=2\pi R$.

PROPOSITION XVI. THEOREM.

492. *The area of a circle is half the product of the circle and its radius.*



Given a ⊙ C, radius R, and area S.

To prove $S = \frac{1}{2} CR.$

Proof. Suppose a regular polygon to be circumscribed about the circle. § 457

Denote its perimeter by P and its area by K.

Then $K = \frac{1}{2} P \times R.$ § 473

Let the number of sides of the polygon be indefinitely increased.

Then K approaches S as a limit,

And since P approaches C as a limit, § 488

$\frac{1}{2} P \times R$ approaches $\frac{1}{2} C \times R$ as a limit. § 277

$\therefore S = \frac{1}{2} C \times R.$ § 275

Q. E. D.

493. COR. 1. *Since area of a circle* $= \frac{1}{2} C \times R$ (§ 492), *and* $C = 2\pi R$ (§ 491), *the area of a circle* $= \frac{1}{2} R \times 2\pi R = \pi R^2$.

494. COR. 2. *The areas of two circles are to each other as the squares of their radii.*

For if S and S' denote the areas, and R and R' the radii,

$$S : S' = \pi R^2 : \pi R'^2 = R^2 : R'^2.$$

495. COR. 3. *The areas of two circles are to each other as the squares of their diameters.*

496. Cor. 4. *The area of a sector is equal to half the product of its arc and radius.*

For a sector and its arc are like parts of a circle and its area.

Therefore, denoting the arc by c and the area by s,

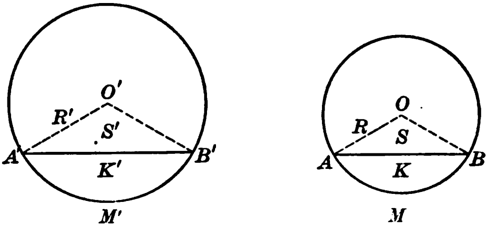
$$\frac{s}{S}=\frac{c}{C}.$$

Whence $$s=c\times\frac{S}{C}=c\times\tfrac{1}{2}R. \qquad \text{§ 492}$$

$$\therefore s=\tfrac{1}{2}cR.$$

497. Def. In circles of different radii, similar arcs, similar sectors, and similar segments are those corresponding to equal central angles.

498. Cor. 5. *The area of similar sectors are to each other as the squares of their radii.*



499. Cor. 6. *The areas of similar segments are to each other as the squares of their radii or as the squares of their chords.*

For let S and S' denote the areas of the similar $\triangle$s AOB and $A'O'B'$, and K and K' the areas of the similar segments AMB and $A'M'B'$.

Then $$S+K:S'+K'=R^2:R'^2. \qquad \text{§ 498}$$

But $$S:S'=R^2:R'^2=\overline{AB}^2:\overline{A'B'}^2. \qquad \text{§ 435}$$

$$\therefore S+K:S=S'+K':S'.$$

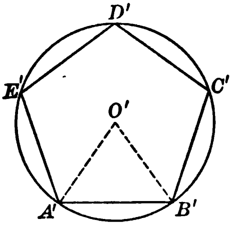
By division, $$K:S=K':S'. \qquad \text{§ 333}$$

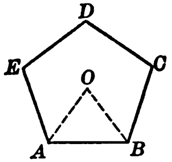
By alternation, $$K:K'=S:S'=R^2:R'^2=\overline{AB}^2:\overline{A'B'}^2. \qquad \text{§ 330}$$

Q. E. D.

Proposition XVII. Problem.

500. *To inscribe in a given circle a regular polygon similar to a given regular polygon.*





Given the regular polygon ABCDE and the $\odot$ O′.

Required to inscribe in circle O' a polygon similar to $ABCDE$.

Construction. From O, the center of the given polygon, draw OA and OB.

From O' draw OA' and OB', making $\measuredangle O' = \measuredangle O$. § 125

Draw $A'B'$.

Then $A'B'$ is a side of the regular polygon required.

Therefore by applying $A'B'$ as a chord, the required polygon $A'B'C'D'E'$ is inscribed.

Proof. Each polygon has as many sides as the number of times $\measuredangle O$, or its equal, $\measuredangle O'$, is contained in a perigon.

$\therefore A'B'C'D'E' \sim ABCDE.$ § 470

Q. E. F.

Ex. 462. Circumscribe a square about a given circle.

Ex. 463. Circumscribe a regular hexagon about a given circle.

Ex. 464. Draw through a given point a line that shall divide a given circle into two parts having the ratio 3 to 5.

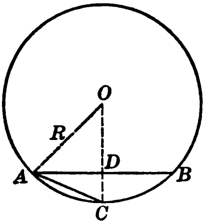
Ex. 465. Construct an angle of 18°.

Ex. 466. Construct an angle of 24°.

Ex. 467. Construct an angle of 42°.

PROPOSITION XVIII. PROBLEM.

501. *Given the radius of a circle and the side of a regular inscribed polygon, to find the side of a regular inscribed polygon of double the number of sides.*



Given the radius OA = R, and AB, the side of a regular inscribed polygon of n sides.

Required to find AC, the side of a regular inscribed polygon of $2n$ sides.

Solution. Draw OA.

Draw OC, meeting AB at D.

Then OC is the ⊥ bisector of AB. § 240

Denote AB, the side of the inscribed polygon, by s.

In rt. △ ADO, $\overline{OD}^2 = \overline{OA}^2 - \overline{AD}^2 = R^2 - \left(\frac{s}{2}\right)^2$. § 392

Whence $OD = \sqrt{R^2 - \frac{s^2}{4}} = \frac{1}{2}\sqrt{4R^2 - s^2}$.

In acute △ AOC, $\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2\,OC \times OD$, § 398

or $\overline{AC}^2 = 2R^2 - 2R \times OD$

$= 2R^2 - R\sqrt{4R^2 - s^2}$.

$\therefore AC = \sqrt{R(2R - \sqrt{4R^2 - s^2})}$. Q. E. F.

502. COR. *If $R = 1$, and the sides of regular inscribed polygons of n sides and $2n$ sides be denoted by S_n and S_{2n}, respectively, then*

$$S_{2n} = \sqrt{2 - \sqrt{4 - S_n^2}}.$$

PROPOSITION XIX. PROBLEM.

503. *To find the numerical value of π, the ratio of the circle to the diameter.*

Let S_6 denote the length of a side of a regular inscribed polygon of 6 sides, S_{12} of 12 sides, etc.

If $R = 1$, $S = 1$. § 476

Then by § 502 we have:

	LENGTH OF SIDE.	LENGTH OF PERIMETER.
$S_{12} = \sqrt{2 - \sqrt{4 - (1)^2}}$	0.51763809	6.21165798
$S_{24} = \sqrt{2 - \sqrt{4 - (0.51763809)^2}}$	0.26105238	6.26525722
$S_{48} = \sqrt{2 - \sqrt{4 - (0.26105238)^2}}$	0.13080626	6.27870041
$S_{96} = \sqrt{2 - \sqrt{4 - (0.13080626)^2}}$	0.06543817	6.28206396
$S_{192} = \sqrt{2 - \sqrt{4 - (0.06543817)^2}}$	0.03272346	6.28290510
$S_{384} = \sqrt{2 - \sqrt{4 - (0.03272346)^2}}$	0.01636228	6.28311544
$S_{768} = \sqrt{2 - \sqrt{4 - (0.01636228)^2}}$	0.00818121	6.28316941

Taking the last perimeter as the approximate value of the circle, $\pi = \frac{C}{2R} = 3.14159$. Q. E. F.

504. NOTE. The approximation 3.1416 is the one most commonly used, and when great accuracy is not required, the value $\pi = 3\frac{1}{7}$ is used.

505. TABLE. The following table of approximate values will be found useful in expediting computations concerning the circle:

$\pi = 3.14159.$

$\frac{\pi}{4} = 0.78540.$

$\frac{1}{\pi} = 0.31831.$

$\pi^2 = 9.86960.$

$\frac{1}{\pi^2} = 0.10132.$

$\sqrt{\pi} = 1.77245.$

$\frac{1}{\sqrt{\pi}} = 0.56419.$

$\sqrt[3]{\pi} = 1.46459.$

506. Historical Note. For many centuries the quadrature of the circle engaged the attention of mathematicians. In plainer language, the problem is: Having given a circle, to construct a square of equal area.

The problem is readily transformable into that of constructing a straight line equal in length to that of a circle having a given diameter, or that of finding the exact ratio of the circle to the diameter.

Mathematicians for a long time thought the problem capable of solution. The Pythagoreans attacked it in vain, and it baffled the genius of Archimedes himself.

An early approximation of π was 3. (I Kings, vii. 23; II Chron. iv. 2.)

Ahmes, an Egyptian mathematician (1700 B.C.), gave the value of π at 3.1604.

Archimedes of Syracuse (287–212 B.C.) found π to be between $3\frac{10}{71}$ and $3\frac{1}{7}$.

Metius (1527–1607) used $\pi = \frac{355}{113}$.

Ptolemy of Alexandria gave 3,1476 as the value.

Ludolph Van Ceulen (1540–1610) computed the value of π to 35 places, a performance at that time extraordinary. On this account it is called by the Germans the "Ludolphian Number."

Vega (1756–1802) carried the value to 140 places; Dase to 200 places; Richter to 500 places; and Shanks, an English mathematician, carried it in 1873 to 707 places.

Lambert of Alsace in 1761 demonstrated the irrationality of π, and finally Lindemann in 1882 proved π to be a transcendental irrational.

On a given line as a side,

Ex. 468. Construct a square.

Ex. 469. Construct a regular pentagon.

Ex. 470. Construct a regular hexagon.

Given the radius of a circle = 10 ft.

Ex. 471. Find the area of an inscribed equilateral triangle.

Ex. 472. Find the area of an inscribed square.

Ex. 473. Find the area of an inscribed regular pentagon.

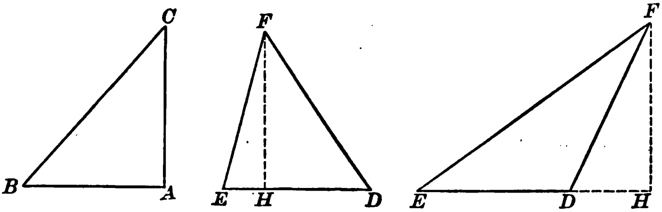
MAXIMA AND MINIMA.

507. Def. A **maximum** is the greatest, and a **minimum** the least of all magnitudes of the same kind.

508. Def. **Isoperimetric** figures are those of equal perimeters.

Proposition XX. Theorem.

509. *Of all triangles having two given sides, that in which these two sides are perpendicular is the maximum.*



Given two $\triangle$s, ABC and DEF, in which AB = DE, AC = DF, $\measuredangle$ A, a rt. $\measuredangle$, and $\measuredangle$ D, an oblique $\measuredangle$.

To prove $\triangle ABC > \triangle DEF$.

Proof. Draw $FH \perp DE$ or DE produced. § 114

Then $FH < DF$. § 116

$\therefore FH < AC$.

$\therefore \triangle ABC > \triangle DEF$. § 427

Q. E. D.

Ex. 474. Find the side of a square equivalent to the sum of three squares whose sides are, respectively, 5 ft., 12 ft., and 84 ft.

Ex. 475. Find the area of the circle whose radius is 12 ft.

Ex. 476. Find the area of the circle whose length is 28 ft.

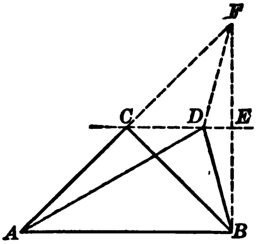
Ex. 477. Find the diameter of the circle whose length is 68 ft.

Ex. 478. The radius of the inscribed circle of an equilateral triangle is half the radius of the circumscribed circle.

Ex. 479. The radius of the inscribed circle of an equilateral triangle is one third the radius of an escribed circle.

PROPOSITION XXI. THEOREM.

510. *Of all equivalent triangles on the same base the isosceles triangle has the least perimeter.*



Given equivalent △s ABC and ABD, in which AC = BC.

To prove the perimeter of △ ABC < the perimeter of △ ABD.

Proof. Draw $CE \parallel AB$. § 154

Then since △s ABC and ABD have the same base and equal areas, they have equal altitudes. § 427

Therefore CE passes through D.

Produce AC to F, making $CF = AC$. Draw FB cutting CE at E, and draw FD.

Because C is the mid-point of AF and CE is $\parallel AB$, Cons.

$$FE = BE. \qquad \S\ 179$$

$$\therefore CE \text{ is the } \perp \text{ bisector of } BF. \qquad \S\ 109$$

Hence

$$DF = BD. \qquad \S\ 108$$

$$AF < AD + DF, \qquad \S\ 86$$

or

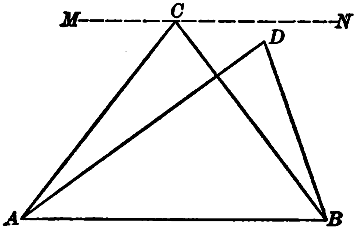
$$AC + BC < AD + BD.$$

∴ perimeter of △ ABC < perimeter of △ ABD. Q. E. D.

511. COR. *Of all triangles of equal areas the equilateral triangle has the least perimeter.*

PROPOSITION XXII. THEOREM.

512. *Of all isoperimetric triangles on the same base, the isosceles triangle is the maximum.*



Given the isosceles △ ABC and the scalene △ ABD, having equal perimeters.

To prove $\triangle ABC > \triangle ABD$.

Proof. Through C draw $MN \parallel AB$. § 154

Then if vertex D fell on MN, the altitudes of the two △s would be equal and, therefore, their areas equal. § 430

Then the perimeter of $\triangle ABD$ would be greater than perimeter of $\triangle ABC$, which is contrary to the hypothesis. § 510

Similarly, if vertex D fell on the opposite side of MN from AB, the perimeter of $\triangle ABD$ would be still greater than perimeter of $\triangle ABC$, which is also contrary to the hypothesis.

$\therefore$ D and AB are on the same side of the line MN; hence the altitude of $\triangle ABD <$ the altitude of $\triangle ABC$.

$\therefore \triangle ABC > \triangle ABD$. § 427

Q. E. D.

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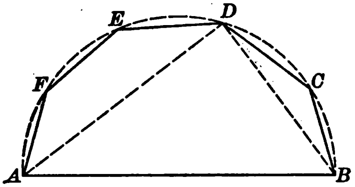
513. COR. *Of all isoperimetric triangles the equilateral triangle is the maximum.*

Ex. 480. The area of an inscribed equilateral triangle is one fourth the area of the circumscribed equilateral triangle.

Ex. 481. The area of an inscribed square is half the area of a circumscribed square.

PROPOSITION XXIII. THEOREM.

514. *Of all polygons having all sides given but one, the maximum can be inscribed in a semicircle having the undetermined side as diameter.*



Given ABCDEF, the maximum of all polygons having the given sides, AF, FE, ED, DC, and CB and AB the undetermined side.

To prove AB is the diameter of the semicircle in which ABCDEF can be inscribed.

Proof. From any vertex D, draw AD and BD.

Then $\triangle ADB$ must be the maximum of all $\triangle$s that can be constructed with the sides AD and BD, for otherwise by increasing or decreasing $\measuredangle ADB$, the $\triangle ABD$ can be increased, while the other parts of the polygon, BCD and $ADEF$ remain unchanged. Thus the area of the polygon $ABCDE$ would be increased, which is contrary to the hypothesis that $ABCDE$ is the maximum. $\therefore ADB$ is a rt. $\measuredangle$. § 509

$\therefore D$ lies on a semicircle whose diameter is AB. § 291

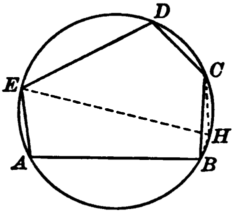
In like manner it is proved that the other vertices, C, E, and F, lie on the semicircle whose diameter is AB. Q. E. D.

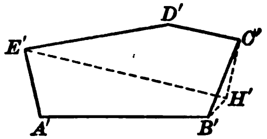
Ex. 482. The area of an inscribed regular hexagon is $\frac{3}{4}$ the area of the circumscribed regular hexagon.

Ex. 483. The area of an inscribed equilateral triangle is half the area of the inscribed regular hexagon.

PROPOSITION XXIV. THEOREM.

515. *Of all polygons constructed with the same given sides, the cyclic polygon is the maximum.*





Given the cyclic polygon ABCDE and the polygon A'B'C'D'E', which has its sides equal, respectively, to the sides of ABCDE, but which is not cyclic.

To prove $ABCDE > A'B'C'D'E'$.

Proof. From any vertex E of $ABCDE$ draw the diameter EH and draw HB and HC.

Upon $B'C'$ construct $\triangle B'C''H' \cong \triangle BCH$ and draw $E'H'$.

Then $EABH > E'A'B'H'$,

and $EHCD > E'H'C'D'$. § 514

Adding, $ABHCDE > A'B'H'C'D'E'$.

But $\triangle BHC \cong B'H'C'$. Cons.

Subtracting, $ABCDE > A'B'C'D'E'$. Ax. 6

Q. E. D.

Ex. 484. The area of an inscribed regular dodecagon is three times the square of the radius of the circle.

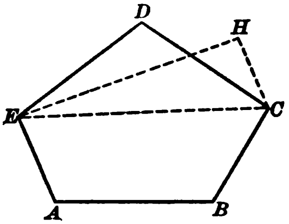
Ex. 485. The square of the side of an inscribed equilateral triangle is three times the square of the side of the inscribed regular hexagon.

Ex. 486. The area of an inscribed regular hexagon is the mean proportional between the areas of the inscribed and the circumscribed equilateral triangle.

Ex. 487. The square of the side of an inscribed equilateral triangle is equal to the sum of the squares of the sides of an inscribed square and inscribed regular hexagon.

PROPOSITION XXV. THEOREM.

516. *Of all isoperimetric polygons of the same number of sides the maximum is equilateral.*



Given ABCD etc., the maximum of isoperimetric polygons of any given number of sides.

To prove *ABCD etc. is equilateral.*

Proof. Draw EC.

The $\triangle CED$ must be the maximum of all isoperimetric $\triangle$s constructed on the base EC, otherwise a $\triangle CHE$ could be substituted for $\triangle CDE$ without changing the perimeter of the polygon.

But this is contrary to the hypothesis that $ABCD$ etc. is the maximum polygon.

$\therefore \triangle CED$ is isosceles. § 512

$\therefore CD = ED.$

Similarly it is proved that any two sides of the polygon are equal.

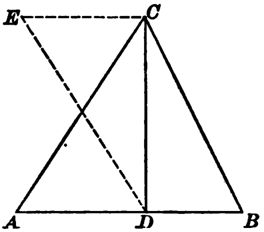
$\therefore ABCD$ etc. is equilateral. Q. E. D.

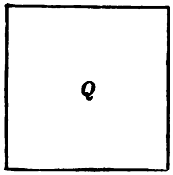
517. COR. *The maximum of isoperimetric polygons of the same number of sides is a regular polygon.*

For the maximum polygon is equilateral (§ 516) and is cyclic (§ 515) and is therefore regular (§ 455).

PROPOSITION XXVI. THEOREM.

518. *Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*





Given ABC a regular polygon of three sides, and Q, a regular polygon of four sides, having its perimeter = that of ABC.

To prove $Q > ABC$.

Proof. Draw CD from C to any point in AB.

Construct $\triangle CED$ having $CE = AD$ and $DE = AC$.

Then $\triangle CED \cong \triangle CAD$. § 123

$\therefore$ quadrilateral $BCED = \triangle ABC$ and has the same perimeter as $\triangle ABC$.

But quadrilateral $BCED < Q$. § 517

$$\therefore Q > ABC.$$

In like manner it is proved that $Q <$ a regular polygon of five sides having the same perimeter, etc. Q. E. D.

519. COR. *The area of a circle is greater than the area of any polygon whose perimeter is equal to the circle.*

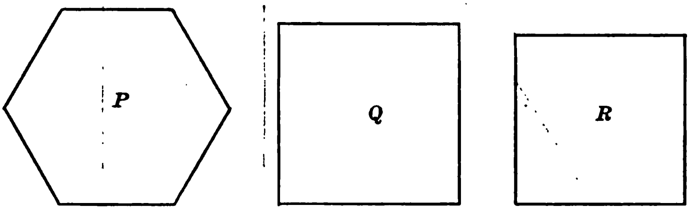
Ex. 488. Circumscribe an equilateral triangle about a given circle.

Ex. 489. Given radius = 10, find the area of an inscribed regular hexagon.

Ex. 490. Given radius = 10, find the area of an inscribed regular dodecagon.

PROPOSITION XXVII. THEOREM.

520. *Of all regular polygons having a given area, that which has the greatest number of sides has the least perimeter.*



Given P and Q, regular polygons having the same area, and P having the greater number of sides.

To prove perimeter of $Q >$ perimeter of P.

Proof. Let R be a regular polygon having the same perimeter as P and the same number of sides as Q.

Then $P > R$. § 518

But $P = Q$. Hyp.

$\therefore Q > R$.

$\therefore$ perimeter of $Q >$ perimeter of R.

But perimeter of $P =$ perimeter of R. Hyp.

$\therefore$ perimeter of $Q >$ perimeter of P. Q. E. D.

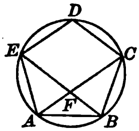
521. COR. *A circle is less than the perimeter of an equivalent polygon.*

Ex. 491. Find the area of an equilateral triangle whose side is 6 ft.

Ex. 492. Find the area of a square whose side is 8 ft.

Ex. 493. Find the area of a regular hexagon whose side is 12 ft.

Ex. 494. Two diagonals of a regular pentagon not drawn from the same vertex divide each other internally in extreme and mean ratio.



$$\triangle ABC \sim \triangle ABF. \quad \therefore AC : AB = AB : AF.$$

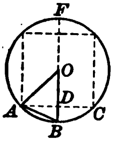
$$\angle CFB = \angle CBF. \quad \therefore CF = BC = AB.$$

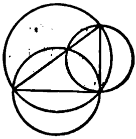
Ex. 495. The greater segment of the diagonal of a regular pentagon, divided internally in extreme and mean ratio, is equal to a side of the pentagon.

Ex. 496. Construct a regular pentagon, having given a diagonal.

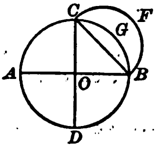
Ex. 497. The area of an inscribed regular octagon is equal to the product of the diameter of the circle by the side of the inscribed square.

$$\text{Area octagon} = 8 \triangle AOB = \frac{8\,(OB \times AD)}{2} = BF \times AC.$$





Ex. 498. The area of the circle on the hypotenuse of a right triangle as diameter is equal to the sum of the areas of the circles on the legs as diameters.



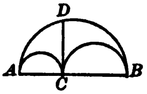
Ex. 499. The area of the figure (crescent) formed by describing a semicircle on the chord of a quadrant is equal in area to the triangle formed by the chord and two radii.

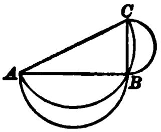
$$\begin{aligned} \text{area} \odot \text{ on } BC &= \text{area} \odot \text{ on } OC + \text{area} \odot \text{ on } OB \text{ (Ex. 498)} \\ &= 2 \text{ area} \odot \text{ on } OC \\ &= \tfrac{1}{2} \text{ area} \odot \text{ on } CD. \end{aligned}$$

$$\therefore \text{area semicircle on } BC = \tfrac{1}{4} \text{ area} \odot \text{ on } CD = \text{quadrant } OBC.$$

From fig. *OBFC* take semicircle on *BC* and from the same figure take quadrant *OBC*.

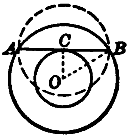
Ex. 500. The sum of the areas of the two crescents formed by semicircles described outwardly on the legs of a right triangle and the semicircle on the hypotenuse is equal to the area of the triangle. (Theorem of Hippocrates.)





Ex. 501. If C is a point on the line AB and semicircles are described with diameters AB, AC, and BC, and CD is drawn perpendicular to AB and terminated by the semicircle on AB, the area bounded by the three semicircles is equal to the area of the circle on CD as diameter.

Ex. 502. If two chords of a circle are perpendicular to each other, the sum of the areas of the four circles on the four segments as diameters is equal to the area of the given circle.



Ex. 503. The area of a circular ring is equal to the area of the circle whose diameter is a chord of the outer and a tangent of the inner circle.

Ex. 504. Construct a circle equal to the sum of two given circles.

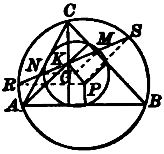
Ex. 505. Construct a circle whose area is equivalent to the sum of the areas of two given circles.

Ex. 506. Construct a circle equal to the difference of two given circles.

Ex 507. Construct a circle whose area is equal to the difference of the areas of two given circles.

Ex. 508. Construct a circle equal to three times a given circle.

Ex. 509. Construct a circle whose area is three times the area of a given circle.



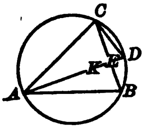
Ex. 510. The nine-point circle of a triangle bisects every line from the orthocenter to the circumscribed circle.

Let K be the orthocenter, P the circumcenter, and G the center of the nine-point circle. Let a line cut the circumscribed ⊙ at R and S, and the nine-point ⊙ at N and M.

$GN = GM = \frac{1}{2} PS$ (Ex. 350). $PG = GK$ (Ex. 351).

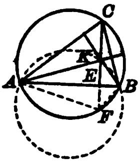
The line from G to mid-point of KS is $\parallel PS$ and $= \frac{1}{2} PS$ and the line from G to mid-point of $KR \parallel PR$ and $= \frac{1}{2} PR$. § 178

∴ N and M are mid-points of KR and KS. § 119



Ex. 511. The line from the vertex of an angle of a triangle to the point in which an altitude to one of its including sides intersects the circumscribed circle is equal to the line from the vertex to the orthocenter.

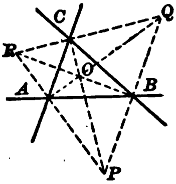
Apply Ex. 312 and Ex. 510 to show that CB is the ⊥ bisector of KD.



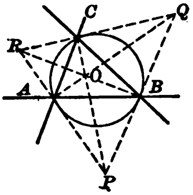
Ex. 512. The three circles, each of which passes through two vertices and the orthocenter of a triangle, are each equal to the circumscribed circle of the triangle.

Prove $\triangle BKA \cong \triangle BFA$.

Ex. 513. Each center of the inscribed circle and an escribed circle of a triangle is the orthocenter of the triangle having the other three points as vertices.



Show that $AQ \perp PR$, $BR \perp PQ$, and $CP \perp QR$.



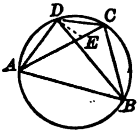
Ex. 514. The circumscribed circle of a triangle is the nine-point circle of each of the four triangles formed by joining the incenter and the excenters of the triangle.

A, B, C, are feet of altitudes of △s ORQ, OPR, PQR, OPQ.

Ex. 515. The circumscribed circle of a triangle bisects the lines between the incenter and each excenter of the triangle, and also the line between any two excenters of the triangle.

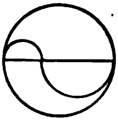
Apply Exs. 514, 513, and 510.

Ex. 516. The four circles, each of which passes through three of the centers of the inscribed and escribed circles of a triangle, are equal.

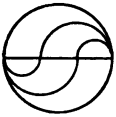


Ex. 517. In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides. (Ptolemy's Theorem.)

Draw DE, making $\measuredangle CDE = \measuredangle ADB$, then $\triangle ABD \sim \triangle CED$ and $\triangle BCD \sim \triangle AED$.

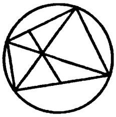


Ex. 518. If the diameter of a circle is divided into any two segments, and upon these segments as diameters semicircles are described upon opposite sides of the diameter, the semicircles divide the area of the circle into parts which have the same ratio as the two segments of the diameter.



Ex. 519. Divide the area of a circle into three equivalent parts by describing semicircles on segments of the diameter.

Ex. 520. Divide the area of a circle into three equivalent parts by concentric circles.



Ex. 521. If the diagonals of a cyclic quadrilateral are perpendicular to each other the line through their point of intersection perpendicular to one side bisects the opposite side.

Ex. 522. Two quadrilaterals are equal in area if their diagonals are respectively equal and make equal angles with each other.

Ex. 523. In the quadrilateral $ABCD$, given $AB = 11$, $BC = 13$, $CD = 15$, $DA = 7$, and $AC = 20$. Find the diagonal BD.

Ex. 524. The sides of a triangle are to each other as $3:4:5$ and the altitude to the longest side is 48 ft. Find the area of the triangle.

Ex. 525. The apothem of a regular inscribed pentagon is equal to half the sum of the radius of the circle and the side of the regular inscribed decagon.

Ex. 526. Compute the side of the square equivalent to the sum of two triangles, the sides of the first being 6 ft., 25 ft., and 29 ft., respectively, and the sides of the second being 8 ft., 29 ft., and 35 ft., respectively.

Ex. 527. One side of triangle is 16 and its perimeter is 96. The homologous side of a similar triangle is 24. What is its perimeter?

Ex. 528. Two polygons, P and Q, are similar. The ratio of their perimeters is 1 : 3. What is the ratio of their areas?

Ex. 529. If the radius of a circle is R and the side of a regular inscribed polygon is a, then the side of a similar circumscribed polygon is $\frac{2aR}{\sqrt{4R^2-a^2}}$.

Ex. 530. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular.

Ex. 531. If two opposite angles of a quadrilateral are right angles, the bisectors of the other angles are parallel.

Ex. 532. Divide a given quadrilateral into two equivalent parts by a line through a vertex.

Ex. 533. Divide a given triangle into two parts having a given ratio by a line drawn parallel to the base.

Ex. 534. If a trapezoid whose bases are 18 and 32, and altitude 28, is divided into two equivalent parts by a line drawn parallel to the bases, what are two segments of the altitude made by such line of division?

Ex. 535. The line joining the mid-points of the bases of a trapezoid passes through the point of intersection of the diagonals.

Ex. 536. The point of intersection of the legs of a trapezoid, when produced, the mid-points of the bases, and the point of intersection of the diagonals are harmonic points.

Denoting the area of a triangle by Δ, the perimeter by $2s$, the sides by a, b, c, the altitudes by h_1, h_2, h_3, the radius of the circumscribed circle by R, the radius of the inscribed circle by r, the radii of the escribed circles by r_1, r_2, r_3:

Ex. 537. Given $a = 12$, $b = 17$, $\Delta = 90$, to find c.

Ex. 538. Given $s = 54$, $\Delta = 234$, $r_1 = 6$, to find a, b, and c.

Ex. 539. Given $a = 17$, $r_2 = 30$, $h_3 = 15$, to find b and c.

Ex. 540. Given $\Delta = 90$, $r_2 = 9$, $r_3 = 45$, to find R.

Ex. 541. Given $s = 16$, $\Delta = 24$, $r_1 = 2$, to find r_3.

Ex. 542. Given $a = 30$, $b = 74$, $c = 88$, to find Δ.

Ex. 543. Given $s = 40$, $\Delta = 180$, $b = 30$, to find R.

Ex. 544. Find the area of the sector of a circle whose central angle is 72°, the radius of the circle being 22 ft.

Ex. 545. If the diameter of a circle be divided into any number of segments and circles be described upon these segments as diameters, the sum of these circles is equal to the original circle.

Ex. 546. Describe circles about the vertices of a given triangle as centers, so that each circle shall be tangent externally to both the others.

Ex. 547. The polygon formed by the diagonals of a regular pentagon is itself a regular pentagon.

Ex. 548. The diameter of a circle is 8 ft. What is the diameter of a circle whose area is 9 times as large?

Ex. 549. How many revolutions will a wheel 4 ft. in diameter make in going 2 mi.?

Ex. 550. Given three equal circles of common radius R, each tangent to the other two. Find the area bounded by the circles.

Ex. 551. The perimeter of a triangle is to one side as the altitude to that side is to the radius of the inscribed circle.

Ex. 552. The area of the square formed by the bisectors of the angles of a rectangle is equal to half the area of the square on the difference of the sides of the rectangle.

Ex. 553. If the corners of a square are cut off so as to leave a regular octagon, the side of this octagon is equal to the difference between the diagonal and the side of the square.

Ex. 554. If AB is the diameter of a circle and the chords AC and BD intersect at P, then $\overline{AB}^2 = AC \times AP + BD \times BP$.

Ex. 555. Construct a square that shall have a given ratio to a given hexagon.

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